

Homework #3 (S.Y. Feb. 2000)

1. Let objects be partially ordered sets and morphisms  $A \xrightarrow{\varphi} B$  be monotonic functions satisfying  $a \leq a' \Rightarrow \varphi(a) \leq \varphi(a')$ .

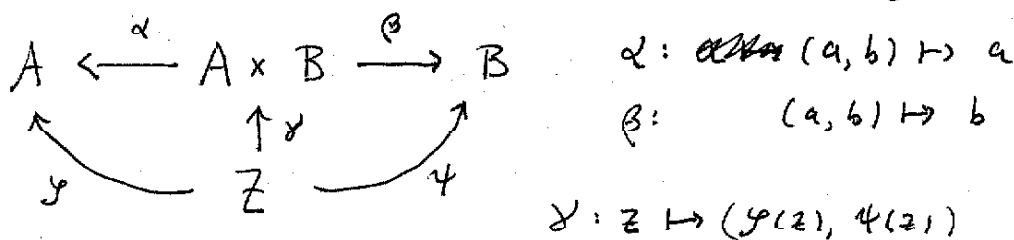
If  $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$  are monotonic, then  $a \leq a' \Rightarrow \varphi(a) \leq \varphi(a') \Rightarrow \psi(\varphi(a)) \leq \psi(\varphi(a')) \Rightarrow \psi \circ \varphi$  is also monotonic.

Since identities are also monotonic and function composition is associative, we have a category.

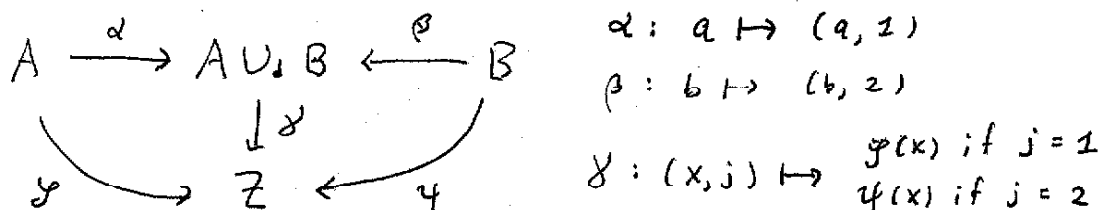
Given  $A$  and  $B$ , define a partial ordering on  $A \times B$  by

$$(a, b) \leq (a', b') \text{ iff } a \leq a' \text{ and } b \leq b'$$

It's easy to check that this is a partial ordering on  $A \times B$ .



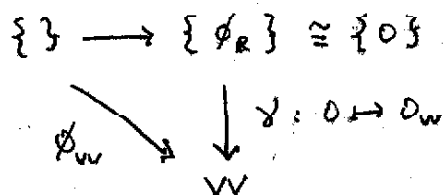
is the direct product and



is the direct sum.

2. Let  $\phi_A: \{\emptyset\} \rightarrow A$  be the empty function from  $\{\emptyset\} \rightarrow A$ .

Then



is the free vector space on the empty set.

3. Given linear  $\varphi: V \rightarrow W$ , suppose that  $\varphi$  is one-to-one but  $X \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\alpha'} \end{matrix} V \xrightarrow{\varphi} W$  commutes for some  $\alpha \neq \alpha'$ . Then  $\alpha(x) \neq \alpha'(x)$  for some  $x \in X \Rightarrow \varphi(\alpha(x)) \neq \varphi(\alpha'(x)) \Rightarrow \Leftarrow$ .

Conversely, suppose that  $\varphi$  is mono but not one-to-one. Then  $\varphi(a) = \varphi(a')$  for some  $a \neq a'$ . Let  $\alpha, \alpha': \mathbb{R} \rightarrow V$  be  $\alpha: r \mapsto r \cdot a$ ,  $\alpha': r \mapsto r \cdot a'$ . Then  $\varphi(\alpha(r)) = \varphi(r \cdot a) = r \varphi(a) = r \varphi(a') = \varphi(r a') = \varphi(\alpha'(r))$  for any  $r \in \mathbb{R}$ . Thus,  $\mathbb{R} \begin{matrix} \xrightarrow{\alpha} \\ \xrightarrow{\alpha'} \end{matrix} V \xrightarrow{\varphi} W$  commutes but  $\alpha \neq \alpha' \Rightarrow \Leftarrow$ .

4. Let  $W'$  be complementary to  $W$ . Then  $V \cong W \oplus W'$ . If  $S \rightarrow W$ ,  $S' \rightarrow W'$  are bases for  $W$  and  $W'$ , then  $S \cup S'$  is a basis for  $W \oplus W'$ . Thus,  $S$  is a subset of a basis for  $V$  ( $S \cup S'$ ) via the direct sum monomorphism  $S \rightarrow S \cup S'$ .

5. Done in the vector space handout.

6. Suppose that  $S \xrightarrow{\alpha} V''$  and  $S' \xrightarrow{\alpha'} V'$  are free and  $S \xrightarrow{\varphi} S'$  is a monomorphism. If  $S$  is not empty,  $\bar{\varphi}_r: S' \rightarrow S$  is a retraction. Let

$$\begin{array}{ccc} \begin{matrix} \mathbb{Q} \text{ is} \\ S \end{matrix} & \xrightarrow{\alpha} & V \\ \bar{\varphi}_r \left( \begin{matrix} \downarrow \varphi \\ S' \end{matrix} \right) & \begin{matrix} \bar{\varphi}_r \uparrow \\ \downarrow \varphi \end{matrix} & \left( \begin{matrix} \downarrow \varphi \\ S' \end{matrix} \right) \end{array} \Rightarrow \bar{\varphi}_r \circ \bar{\varphi} \circ \alpha = \alpha \Rightarrow \bar{\varphi}_r \text{ is a retraction of } \bar{\varphi} \\ \Rightarrow \bar{\varphi} \text{ is mono.} \end{array}$$

Similarly, if  $S \xrightarrow{\varphi} S'$  is epi, there is an epimorphism from  $V$  to  $V'$ .

Conversely, if there is a mono  $V \xrightarrow{f} V'$ , then  $V \cong \text{Im } f$   
and  $S$  is a subset of some basis of  $V'$  (problem 4).  
The injection of  $S$  into this basis is a monomorphism.

Dually, if there is an epi  $V \xrightarrow{g} V'$ , then  $V/\text{Ker } g \cong V'$   
and  $V \cong \text{Ker } g \oplus V/\text{Ker } g \Rightarrow$  there is an epimorphism  
from  $S$  to  $S'$ .