Homework #3 (S.Y. Feb. 2000)

1. Let objects be partially ordered sets and morphisms $A \xrightarrow{\varphi} B$ be <u>monotonic</u> functions satisfying $a \le a' \Rightarrow \varphi(a) \le \varphi(a')$.

If $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ are monotonic, then $a \le a' \Rightarrow \varphi(a) \le \varphi(a')$ $\Rightarrow \psi(\varphi(a)) \le \psi(\varphi(a')) \Rightarrow \psi \circ \varphi$ is also monotonic.
Since identities are also monotonic and function composition is associative, we have a category.

Given A and B, define a partial ordering on $A \times B$ by

$$(a,b) \le (a',b') \quad \text{iff} \quad a \le a' \text{ and } b \le b'$$

It's easy to check that this is a partial ordering on $A \times B$.

$$\begin{tikzcd} A & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] & B \\ & Z \arrow[ul, bend left, "\varphi"] \arrow[u, "\gamma"] \arrow[ur, bend right, "\psi"'] & \end{tikzcd}$$

$\alpha : (a,b) \mapsto a$
$\beta : (a,b) \mapsto b$

$\gamma : z \mapsto (\varphi(z), \psi(z))$

is the direct product and

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[dr, bend right, "\varphi"'] & A \cup_d B \arrow[d, "\gamma"] & B \arrow[l, "\beta"'] \arrow[dl, bend left, "\psi"] \\ & Z & \end{tikzcd}$$

$\alpha : a \mapsto (a, 1)$
$\beta : b \mapsto (b, 2)$
$\gamma : (x, j) \mapsto \begin{cases} \varphi(x) & \text{if } j = 1 \\ \psi(x) & \text{if } j = 2 \end{cases}$

is the direct sum.

2. Let $\emptyset_A : \{\} \to A$ be the empty function from $\{\} \to A$.
Then

$$\begin{tikzcd} \{\} \arrow[r] \arrow[dr, "\emptyset_W"'] & \{\emptyset_{\mathbb{R}}\} \cong \{0\} \arrow[d, "\gamma : 0 \mapsto 0_W"] \\ & W \end{tikzcd}$$

is the free vector space on the empty set.

3. Given linear $\varphi : V \to W$, suppose that φ is one-to-one but $X \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} V \xrightarrow{\varphi} W$ commutes for some $\alpha \neq \alpha'$. Then $\alpha(x) \neq \alpha'(x)$ for some $x \in X \Rightarrow \varphi(\alpha(x)) \neq \varphi(\alpha'(x)) \Rightarrow \Leftarrow$.

Conversely, suppose that φ is mono but not one-to-one. Then $\varphi(a) = \varphi(a')$ for some $a \neq a'$. Let $\alpha, \alpha' : \mathbb{R} \to V$ be $\alpha : r \mapsto r \cdot a$, $\alpha' : r \mapsto r \cdot a'$. Then $\varphi(\alpha(r)) = \varphi(r \cdot a) = r\varphi(a) = r\varphi(a') = \varphi(ra') = \varphi(\alpha'(r))$ for any $r \in \mathbb{R}$. Thus, $\mathbb{R} \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} V \xrightarrow{\varphi} W$ commutes but $\alpha \neq \alpha' \Rightarrow \Leftarrow$.

4. Let W' be complementary to W. Then $V \cong W \oplus W'$. If $S \to W$, $S' \to W'$ are bases for W and W', then $S \cup_d S'$ is a basis for $W \oplus W'$. Thus, S is a subset of a basis for V ($S \cup_d S'$) via the direct sum monomorphism $S \to S \cup_d S'$.

5. Done in the vector space handout.

6. Suppose that $S \xrightarrow{\alpha} V$ and $S' \xrightarrow{\alpha'} V'$ are free and $S \xrightarrow{\varphi} S'$ is a monomorphism. If S is not empty, $\varphi_r : S' \to S$ is a retraction. Let

$$\begin{tikzcd} S \arrow[loop above, "i_S"] \arrow[r, "\alpha"] \arrow[d, shift left, "\varphi"] & V \arrow[d, shift left, "\bar\varphi"] \\ S' \arrow[u, bend left=50, "\varphi_r"] \arrow[r, "\alpha'"] & V' \arrow[u, shift left, "\bar\varphi_r"] \end{tikzcd}$$

where $\bar\varphi$ and $\bar\varphi_r$ are the induced morphisms.

$\Rightarrow \bar\varphi_r \circ \bar\varphi \circ \alpha = \alpha \Rightarrow \bar\varphi_r$ is a retraction of $\bar\varphi$

$\Rightarrow \bar\varphi$ is mono.

Similarly, if $S \xrightarrow{\varphi} S'$ is epi, there is an epimorphism from V to V'.

Conversely, if there is a mono $V \xrightarrow{\varphi} V'$, then $V \cong \operatorname{Im} \varphi$ and S is a subset of some basis of V' (problem 4). The injection of S into this basis is a monomorphism.

Dually, if there is an epi $V \xrightarrow{\varphi} V'$, then $V/\operatorname{Ker} \varphi \cong V'$ and $V \cong \operatorname{Ker} \varphi \oplus V/\operatorname{Ker} \varphi$ $\Rightarrow$ there is an epimorphism from S to S'.