Homework #2 (S.Y.)

1. $g'^{-1}g^{-1}(gg') = (gg')g'^{-1}g^{-1} = e$

2. No, because 0 has no inverse. Let $\mathbb{R}$ be the additive group of reals and let $\mathbb{R}^+$ be the multiplicative group of nonnegative reals. Then

$$\begin{tikzcd} \mathbb{R} \arrow[loop left, "i_{\mathbb{R}}"] \arrow[r, bend left, "\exp"] & \mathbb{R}^+ \arrow[l, bend left, "\log"] \arrow[loop right, "i_{\mathbb{R}^+}"] \end{tikzcd}$$

commutes and $\exp$ is a morphism because $\exp(r+r') = \exp(r)\exp(r')$. Similarly, $\log(rr') = \log r + \log r'$ for $r, r' \in \mathbb{R}^+$.

3. The integers mod n satisfies this.

4. No, for example, in $\mathbb{R}^2$, the x ~~and~~ axis and y axis are subgroups, but their union is not closed under "+".

5.
$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "f"'] & G \arrow[d, "\varphi_f"] \\ S' \arrow[r, "\alpha'"] & G' \end{tikzcd}$$

6. Hardly any. This is unlikely since $\mathrm{Perm}(G)$ is so huge. To prove this, let $f: G \to \mathrm{Perm}(G)$ have the action

$$f: g \mapsto \text{the permutation that swaps } e \text{ and } g.$$

f is clearly one-to-one. Suppose that G has at least 2 elements besides e. Let g and g' be such. But then the permutation that swaps g and g' is not in $\mathrm{Im}\, f$ $\Rightarrow G \not\cong \mathrm{Perm}\, G$ unless $|G| = 1$ or 2. If $|G| = 1$ or 2, $G \cong \mathrm{Perm}\, G$.

7. No, because inverses are lacking.

8. Find the free Abelian group on a set.

I'll do this two different ways, one more explicit and the other more abstract.

Explicit:

A finite bag (FB) is a set (finite) where duplicate elements are allowed. Define

$$\{a_1, a_2, \dots, a_n\} \cup \{b_1, b_2, \dots, b_m\} \equiv \{a_1, a_2, \dots, a_n, b_1, b_2, \dots b_m\}.$$

Let "$A \subset B$" if $a \in B$ for every $a \in A$. Let FBs $A = B$ if $A \subset B$ and $B \subset A$.

Given a set S, let S' denote "elements of S with primes attached".

Let A (the free abelian group on S) be the collection of FBs whose elements are in $S \cup S'$ and which contains no sub FBs $\{s, s'\}$ for $s \in S$. Let $\{\dots\}_R$ denote removing any $\{s, s'\}$ pairs from an FB and let $\{\dots\}_I$ denote replacing all primes with no-primes and no-primes with primes. Define the group operation on A to be

$$a + b \equiv (a \cup b)_R$$

Then this is associative $[((a \cup b)_R \cup c)_R = (a \cup (b \cup c)_R)_R = (a \cup b \cup c)_R]$, $\{\}$ is the identity, and a_I is the inverse of $a \in A$. Thus, A is an Abelian group.

I claim that $S \xrightarrow{\alpha} A$ is a free abelian group where $\alpha : s \mapsto \{s\}$.

To prove this, I must show that for any abelian group B and any $f: S \to B$, there exists a unique γ s.t.

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "f"'] & A \arrow[d, "\gamma"] \\ & B \end{tikzcd}$$

group homomorphism.

commutes.

For the diagram to commute, we must clearly have

$$\gamma : \{s\} \mapsto f(s).$$

But if γ is a group homomorphism, then this also uniquely determines $\gamma(\{s_1, s_2', \dots, s_n'\})$ since

$\gamma(\{s_1, s_2', \dots, s_n'\}) = \gamma(\{s_1\} + \{s_2'\} + \dots + \{s_n'\}) =$
$\gamma(\{s_1\})\gamma(\{s_2'\}) \cdots \gamma(\{s_n'\}) = f(s_1) f(s_2') \cdots f(s_n')$

and $\gamma(\{s\} + \{s'\}) = f(s) f(s') = \gamma(\{\}) = e_B$
$\Rightarrow f(s') = f(s)^{-1}$. $\Rightarrow \gamma(\{\dots\}_R) = \gamma(\{\dots\})$. Thus, we have $\gamma(a+b) = \gamma((a \cup b)_R) = \gamma(a \cup b) = \gamma(a)\gamma(b)$ and γ is a group homomorphism.

<u>Implicit</u> Let $S \xrightarrow{\alpha} G$ be the free group on S and let C be G's commutator subgroup (see Geroch). Consider

natural insertion

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[dr, "f"'] & G \arrow[r, "\beta"] \arrow[d, "\gamma"] & G/C \arrow[dl, "\gamma'"] \\ & A & \end{tikzcd}$$

By the freeness of G, there is a unique γ causing the left triangle to commute.

Since $\gamma(c) = e_A$ for any $c \in C$, $C \subset \operatorname{Ker} \gamma$ and there is a unique γ' s.t. the right triangle commutes.
Thus $(\beta \circ \alpha, G/C)$ is the free abelian group on S.

10. Let $\mathbb{R}$ be the additive group of reals and $\mathbb{Z}$ be the additive group of integers. $\mathbb{Z}$ is normal in $\mathbb{R}$, so we can form

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\alpha"] \arrow[dr, "\text{fraction}"'] & \mathbb{R}/\mathbb{Z} \arrow[d, "\gamma"] \\ & {[0,1)} \end{tikzcd} \qquad \alpha : r \mapsto r + \mathbb{Z}$$

Additive group of reals modulo 1.

where fraction (r) is the fractional part of real r.
Note that fraction is an epimorphism and $\mathrm{Ker}(\text{fraction}) = \mathbb{Z}$.
$\Rightarrow \mathbb{R}/\mathbb{Z} \cong [0,1)$.

~~11. First, it helps to notice that the left cosets~~
~~$L = \{gH$~~

11. Suppose that H is a subgroup of G. It helps to notice that the left cosets $L = \{gH : g \in G\}$ and the right cosets $R = \{Hg : g \in G\}$ are isomorphic as sets $[f : L \to R,\ f : gH \mapsto Hg^{-1}$ is an isomorphism $]$. Thus, if there are two left cosets $\{H, G-H\}$, then there are also two right cosets $\{H, G-H\} \Rightarrow H$ is normal.

12. Given $G \xrightarrow{\varphi} H$, the onto part is true by definition. Suppose that $\mathrm{Ker}\,\varphi = \{e\}$. Then $\varphi(g) = \varphi(g') \Rightarrow \varphi(gg'^{-1}) = e$ $\Rightarrow gg'^{-1} = e \Rightarrow g = g' \Rightarrow \varphi$ is one-to-one. On the other hand, if φ is one-to-one, $\varphi(e) = e$ and if, for any g, $\varphi(g) = e$, $g = e$ $\Rightarrow \mathrm{Ker}\,\varphi = \{e\}$.

13. For $\varphi, \psi \in \mathrm{Mor}(G, H)$, we can define

$\psi * \varphi : g \mapsto \psi(g)\varphi(g)$, but unless H is abelian, this is not generally a group homomorphism.

14. Consider the following diagram for groups G, H

$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[d, leftrightarrow, "i_G"'] & G \cup_d H \arrow[d, "\lambda"] \arrow[dd, dashed, bend left] & H \arrow[l, "\beta"'] \arrow[d, leftrightarrow, "i_H"] \\
G \arrow[r, "\lambda\circ\alpha"] \arrow[dr, bend right, "\varphi"'] & F \arrow[d, "\gamma'"] & H \arrow[l, "\lambda\circ\beta"'] \arrow[dl, bend left, "\psi"] \\
& Z &
\end{tikzcd}$$

$G \cup_d H \xrightarrow{\lambda} F$ the free group on $G \cup_d H$

Given any φ, ψ, Z, by the set-direct-sum, there is a unique function γ s.t. the outer rim commutes. By the freeness of $G \cup_d H \xrightarrow{\lambda} F$, there is a unique morphism γ' s.t. the free triangle commutes. Thus, we have

$\gamma' \circ \lambda = \gamma$, $\gamma \circ \alpha = \varphi$, $\gamma \circ \beta = \psi$

$\Rightarrow \gamma' \circ (\lambda \circ \alpha) = \varphi$ and $\gamma' \circ (\lambda \circ \beta) = \psi$

$\Rightarrow (F, \lambda\circ\alpha, \lambda\circ\beta)$ is the direct sum of G and H.

15. Consider
$$\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[dr, bend right, "e"'] & G \times H \arrow[d, "\gamma"] & H \arrow[l, "\beta"'] \arrow[dl, bend left, "i_H"] \\
& H &
\end{tikzcd}$$
$e : g \mapsto e_H$

$\gamma : (g, h) \mapsto h$

Then $\mathrm{Ker}(\gamma) = \{(g, e_H) : g \in G\} \cong G \Rightarrow G$ is normal in $G \times H$.

Since γ is epi,

$$\begin{tikzcd}
G \times H \arrow[r] \arrow[dr, "\gamma"'] & G \times H / G \arrow[d, "\gamma'"] \\
& H
\end{tikzcd}$$

γ' is an isomorphism

$\Rightarrow G \times H / G \cong H$.