

Homework #2 (S.Y.)

1. $g^{-1}g^{-1}(gg') = (gg')g^{-1}g^{-1} = e$

2. No, because 0 has no inverse. Let \mathbb{R} be the additive group of reals and let \mathbb{R}^+ be the multiplicative group of nonnegative reals. Then

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\exp} & \mathbb{R}^+ \cong i_{\mathbb{R}^+} \\ \downarrow i_{\mathbb{R}} & & \downarrow \log \\ \mathbb{R} & \xrightarrow{\log} & \mathbb{R}^+ \end{array}$$

commutes and \exp is a morphism because $\exp(r+r') = \exp(r)\exp(r')$. Similarly, $\log(rr') = \log r + \log r'$ for $r, r' \in \mathbb{R}^+$.

3. The integers mod n satisfies this.

4. No, for example, in \mathbb{R}^2 , the x axis and y axis are subgroups, but their union is not closed under "+".

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & G \\ f \downarrow & & \downarrow \varphi_f \\ S' & \xrightarrow{\alpha'} & G' \end{array}$$

6. Hardly any. This is unlikely since $\text{Perm}(G)$ is so huge. To sketch prove this, let $f: G \rightarrow \text{Perm}(G)$ have the action

$f: g \mapsto$ the permutation that swaps e and g .

f is clearly one-to-one. Suppose that G has at least 2 elements besides e . Let g and g' be such. But then the permutation that swaps g and g' is not in $\text{Im } f$

$\Rightarrow G \not\cong \text{Perm } G$ unless $|G| = 1$ or 2 . If $|G| = 1$ or 2 ,

$G \cong \text{Perm } G$.

7. No, because inverses are lacking.

8. Find the free Abelian group on a set.

I'll do this two different ways, one more explicit and the other more abstract.

Explicit:

A finite bag (FB) is a set (finite) where duplicate elements are allowed. Define

$$\{a_1, a_2, \dots, a_n\} \cup \{b_1, b_2, \dots, b_m\} \equiv \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m\}.$$

Let " $A \subset B$ " if $a \in B$ for every $a \in A$. Let FBs $A = B$ if $A \subset B$ and $B \subset A$.

Given a set S , let S' denote elements of S with primes attached.

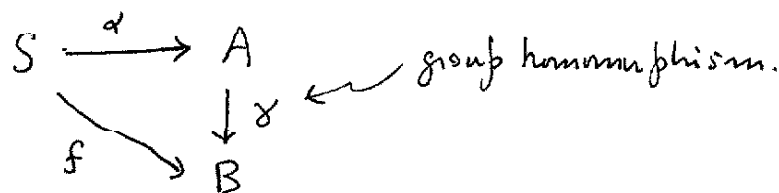
Let A (the free abelian group on S) be the collection of FBs whose elements are in $S \cup S'$ and which contains no sub FBs $\{s, s'\}$ for $s \in S$. Let $\{\dots\}_R$ denote removing any $\{s, s'\}$ pairs from an FB and let $\{\dots\}_I$ denote replacing all primes with no-primes and no-primes with primes. Define the group operation on A to be

$$a + b \equiv (a \cup b)_R$$

Then this is associative $[(a \cup b)_R \cup c]_R = (a \cup (b \cup c)_R)_R = (a \cup b \cup c)_R$, $\{\}$ is the identity, and a_I is the inverse of $a \in A$. Thus, A is an Abelian group.

I claim that $S \xrightarrow{\alpha} A$ is a free abelian group where $\alpha: s \mapsto \{s\}$.

To prove this, I must show that for any abelian group B and any $f: S \rightarrow B$, there exists a unique γ s.t.



commutes.

For the diagram to commute, we must clearly have

$$\gamma: \{s\} \mapsto f(s).$$

But if γ is a group homomorphism, then this also uniquely determines $\gamma(\{s_1, s_2, \dots, s_n\})$ since

$$\begin{aligned}
 \gamma(\{s_1, s_2, \dots, s_n\}) &= \gamma(\{s_1\} + \{s_2\} + \dots + \{s_n\}) = \\
 \gamma(\{s_1\})\gamma(\{s_2\})\dots\gamma(\{s_n\}) &= f(s_1)f(s_2)\dots f(s_n)
 \end{aligned}$$

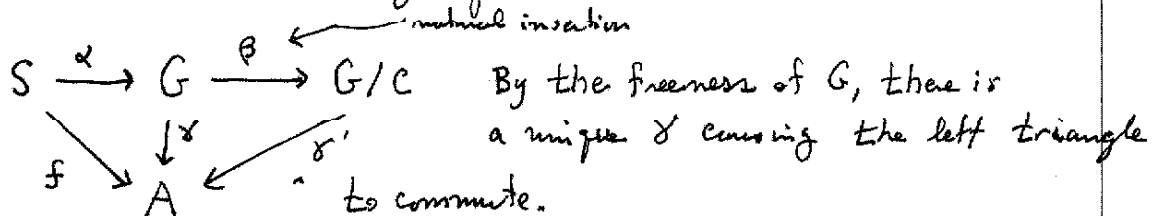
$$\text{and } \gamma(\{s\} + \{s'\}) = f(s)f(s') = \gamma(\{s\}) = e_B$$

$$\Rightarrow f(s') = f(s)^{-1} \Rightarrow \gamma(\{\dots\}_R) = \gamma(\{\dots\}). \text{ Thus, we}$$

have $\gamma(a+b) = \gamma((a \cup b)_R) = \gamma(a \cup b) = \gamma(a)\gamma(b)$ and

γ is a group homomorphism.

Implicit Let $S \xrightarrow{\alpha} G$ be the free group on S and let C be G 's commutator subgroup (see Geroch). Consider



Since $\gamma(c) = e_A$ for any $c \in C$, $C \subset \text{Ker } \gamma$ and there is a unique γ' s.t. the right triangle commutes.

Thus $(\beta \circ \alpha, G/C)$ is the free abelian group on S .

10. Let \mathbb{R} be the additive group of reals and \mathbb{Z} be the additive group of integers. \mathbb{Z} is normal in \mathbb{R} , so we can form

$$\begin{array}{ccc}
 \mathbb{R} & \xrightarrow{\alpha} & \mathbb{R}/\mathbb{Z} & \alpha: r \mapsto r + \mathbb{Z} \\
 \searrow \text{fraction} & & \downarrow \delta & \\
 & & [0, 1) & \leftarrow \text{Additive group of reals} \\
 & & & \text{modulo 1.}
 \end{array}$$

where $\text{fraction}(r)$ is the fractional part of real r .
 Note that fraction is an epimorphism and $\text{Ker}(\text{fraction}) = \mathbb{Z}$.
 $\Rightarrow \mathbb{R}/\mathbb{Z} \cong [0, 1)$.

~~11. First, it helps to notice that the left cosets~~
 ~~$L = \{gH\}$~~

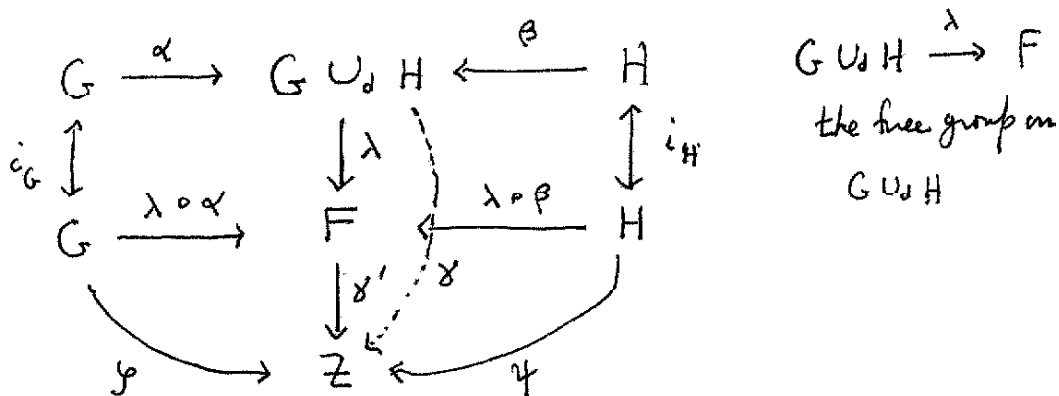
11. Suppose that H is a subgroup of G . It helps to notice that the left cosets $L = \{gH : g \in G\}$ and the right cosets $R = \{Hg : g \in G\}$ are isomorphic as sets [$f: L \rightarrow R, f: gH \mapsto Hg^{-1}$ is an isomorphism]. Thus, if there are two left cosets $\{H, G-H\}$, then there are also two right cosets $\{H, G-H\} \Rightarrow H$ is normal.

12. Given $G \xrightarrow{y} H$, the onto part is true by definition. Suppose that $\text{Ker } y = \{e\}$. Then $y(g) = y(g') \Rightarrow y(gg'^{-1}) = e \Rightarrow gg'^{-1} = e \Rightarrow g = g' \Rightarrow y$ is one-to-one. On the other hand, if y is one-to-one, $y(e) = e$ and if, for any g , $y(g) = e$, $g = e \Rightarrow \text{Ker } y = \{e\}$.

13. For $\varphi, \psi \in \text{Map}(G, H)$, we can define

$\psi * \varphi : g \mapsto \psi(g)\varphi(g)$, but unless H is abelian, this is not generally a group homomorphism.

14. Consider the following diagram for groups G, H

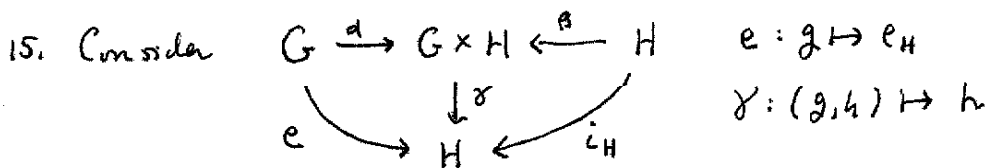


Given any φ, ψ, z , by the set-direct-sum, there is a unique function γ s.t. the outer rim commutes. By the freeness of $G \cup_d H \xrightarrow{\lambda} F$, there is a unique morphism γ' s.t. the free triangle commutes. Thus, we have

$$\gamma' \circ \lambda = \gamma, \quad \gamma \circ \alpha = \varphi, \quad \gamma \circ \beta = \psi$$

$$\Rightarrow \gamma' \circ (\lambda \circ \alpha) = \varphi \text{ and } \gamma' \circ (\lambda \circ \beta) = \psi$$

$\Rightarrow (F, \lambda \circ \alpha, \lambda \circ \beta)$ is the ~~free~~ direct sum of G and H .



Then $\text{Ker}(\gamma) = \{(g, e_H) : g \in G\} \cong G \Rightarrow G$ is normal in $G \times H$.

Since γ is epi,

$$\begin{array}{ccc}
 G \times H & \longrightarrow & G \times H / G \\
 \searrow \gamma & & \downarrow \gamma' \\
 & & H
 \end{array}$$

γ' is an isomorphism
 $\Rightarrow G \times H / G \cong H$.