Homework # 1 Solutions. (S.Y.)

1. Two functions are equal if they have the same domain and codomain and if they have the same value at each point in their domain. Sets are equal if they have the same elements.

2. If set A has n elements and set B has m elements then there are m^n functions from A to B. There are two functions from $\{1\}$ to $\{a,b\}$, one (the empty function) from $\{\}$ to $\{a,b\}$ and none from $\{1,2\}$ to $\{\}$.

3. The problem is that f is not necessarily a function. We might have $[x]=[x']$ with $x \neq x'$, so f would not define a function from X/E to X.

4. Objects are typically sets, but not always. Morphisms are typically functions but not always. See, e.g. # 10.

5. Suppose that $A \xrightarrow{\varphi} B$ is an isomorphism and

$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\alpha'"'] & X \end{tikzcd}$$

commutes for some α, α', X. Then

$$\alpha \circ \varphi = \alpha' \circ \varphi \Rightarrow \alpha \circ (\varphi \circ \varphi^{-1}) = \alpha' \circ (\varphi \circ \varphi^{-1}) \Rightarrow \alpha = \alpha'$$

$\Rightarrow \varphi$ is an epimorphism. Similarly, if

$$\begin{tikzcd} X \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\alpha'"'] & A \arrow[r, "\varphi"] & B \end{tikzcd}$$

commutes, $\varphi \circ \alpha = \varphi \circ \alpha' \Rightarrow \varphi^{-1} \circ \varphi \circ \alpha = \varphi^{-1} \circ \varphi \circ \alpha'$

$\Rightarrow \alpha = \alpha' \Rightarrow \varphi$ is a monomorphism.

6. Suppose that isomorphism $A \xrightarrow{\varphi} B$ has two inverses ψ_1 and ψ_2. Then $i_A = \psi_1 \circ \varphi = \psi_2 \circ \varphi \Rightarrow \psi_1 \circ (\varphi \circ \psi_1) = \psi_2 \circ (\varphi \circ \psi_2)$ $\Rightarrow \psi_1 \circ i_B = \psi_2 \circ i_B \Rightarrow \psi_1 = \psi_2$.

7. Suppose that $f : A \to B$ is one-to-one and A is nonempty. Choose any $a^* \in A$ and let

$$f_r : b \mapsto \begin{cases} a & \text{if } b = f(a) \\ a^* & \text{otherwise.} \end{cases}$$

You can easily check that f_r is a function and that $f_r(f(a)) = a$ for each $a \in A$. Conversely, suppose that f has a retraction f_r. Then $f(a) = f(a') \Rightarrow f_r(f(a)) = f_r(f(a'))$ $\Rightarrow a = a' \Rightarrow f$ is one-to-one.

Suppose that $f : A \to B$ has a section $f_s : B \to A$, and suppose that f is not onto. Then there must be a $b \in B$ s.t. $f(a) \neq b$ for every $a \in A$. Choose $a = f_s(b) \Rightarrow \Leftarrow$. Conversely, suppose that f is onto, then

$$f_s : b \mapsto \text{any } a \in A \text{ s.t. } f(a) = b$$

is a section.

8. Suppose that $\psi \circ \varphi$ is mono and we have an X, $\alpha \neq \alpha'$ s.t. $X \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} A \xrightarrow{\varphi} B$ commutes. But then $X \underset{\alpha'}{\overset{\alpha}{\rightrightarrows}} A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ also commutes $\Rightarrow \alpha = \alpha' \Rightarrow \Leftarrow$. Similarly for epi.

9. This isn't true in general. Consider a category containing only two morphisms like so

$$B \longleftarrow A \longrightarrow C$$

plus the identities. Then A is vacuously a direct product of B and C independent of the morphisms.

On the other hand, suppose that A and B have a direct product and there is some morphism $A \xrightarrow{\varphi} B$. Then

$$\begin{tikzcd}
A & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] & B \\
& A \arrow[u, "\gamma"] \arrow[ul, bend left, "i_A"] \arrow[ur, bend right, "\varphi"'] &
\end{tikzcd}$$

commutes for a unique γ. Since $\alpha \circ \gamma = i_A$, it is an isomorphism $\Rightarrow$ an epimorphism $\Rightarrow$ α is epi, by problem 5. Similarly, if there is some morphism $A \xleftarrow{\psi} B$, β is epi.

Also, if A and B have a direct sum,

$$\begin{tikzcd}
A \arrow[r, "\alpha'"] \arrow[dr, bend right, "i_A"'] & A \oplus B \arrow[d, "\gamma'"] & B \arrow[l, "\beta'"'] \arrow[dl, bend left, "\psi"] \\
& A &
\end{tikzcd}$$

$\gamma' \circ \alpha'$ is an isomorphism $\to$ α' is mono. Similarly for β'.

10. Suppose that we have two categories C and C'. Let objects in the "product" be pairs (A, A') where A is an object in C and A' is an object in C'. Similarly, let morphisms be pairs (φ, φ') where $\varphi \in C$, $\varphi' \in C'$. Let the domain of (φ, φ') be $(\mathrm{dom}\,\varphi, \mathrm{dom}\,\varphi')$, let the codomain be $(\mathrm{codom}\,\varphi, \mathrm{codom}\,\varphi')$.

Let composition be defined by

$$\begin{tikzcd}(A, A') \arrow[r, "(\varphi, \varphi')"] \arrow[rr, bend right, "(\psi\circ\varphi,\ \psi'\circ\varphi')"'] & (B, B') \arrow[r, "(\psi, \psi')"] & (D, D')\end{tikzcd}$$

let the identity of (A, A') be $(i_A, i_{A'})$.

It's easy to check that composition is associative and that identities have the right properties.
Thus, we have a category.

11. It's easy to verify that "17" is a category.
Observe that in "17" every epimorphism is also an isomorphism.
Then if A and B have a direct product, for any φ, ψ,

$$\begin{tikzcd} A & A \times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] & B \\ & X \arrow[ul, bend left, "\varphi"] \arrow[u, "\gamma"] \arrow[ur, bend right, "\psi"'] & \end{tikzcd}$$

commutes for a unique γ. But α is epi $\Rightarrow$ iso
$\Rightarrow \alpha\circ\gamma = \varphi \Rightarrow \gamma = \alpha^{-1}\circ\varphi = \beta^{-1}\circ\psi \Rightarrow (\beta\circ\alpha^{-1})\circ\varphi = \psi$
choose φ iso and ψ not iso to reach a contradiction.
Similarly for direct sum. $\Rightarrow$ no direct sums or products exist in "17".

12. Problem 7 uses the "Axiom of Choice".