Homework 1
Mathematical Physics
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1. How can you tell when two functions are equal? How can you tell when two sets are equal?

2. How many functions are there between $\{1, 2, 3\}$ and $\{a, b\}$? How many between $\{1\}$ and $\{a, b\}$? Between $\{\}$ and $\{a, b\}$? Between $\{1, 2\}$ and $\{\}$? What's the general formula?

3. Consider an equivalence relation E on set X. Let f be the function mapping $[x]$ to x. What's wrong with this?

4. Do you remember the definition of a category? What's an "object" anyway? What's a morphism? Is a morphism always a function? Is an object always a set?

5. (Geroch's Exercise 1): Prove that every isomorphism is both a monomorphism and an epimorphism.

6. (Geroch's Exercise 5): Prove the the inverse of an isomorphism is unique.

7. A function $f : A \to B$ is said to have a *retraction* if a function $g : B \to A$ exists such that $g \circ f$ is the identity and f is said to have a *section* if there exists an $h : B \to A$ such that $f \circ h$ is the identity. Prove that if A is nonempty, f has a retraction if and only if f is one-to-one. Prove that f has a section if and only if f is onto.

8. (Geroch's Exercise 2): Let $A \to^{\phi} B \to^{\psi} C$. Prove that if $\psi \circ \phi$ is a monomorphism, so is ϕ and if $\psi \circ \phi$ is an epimorphism, so is ψ. Find examples in the category of sets showing that the converses of these two statements are false.

9. (Geroch's Exercise 7): In the category of sets, the two projections in the definition of a direct product are epimorphisms and the two insertions in the direct sum are monomorphisms [Geroch's statement of Exercise 7 has epi and mono incorrectly switched.]. Is this true in every category?

10. (Geroch's Exercise 11): Fix two categories. Introduce a new category that can be though of as the "product" of these. (Hint: choose, for objects, pairs consisting of one object from each of the given categories).

11. (Geroch's Exercise 4): Let the objects be sets with exactly 17 elements, the morphisms mappings of such sets, and composition composition. Verify that this is a category. Prove that in this category, no two objects have either a direct product or a direct sum.

12. [Extra credit] The solution to one of the above problems relies on a mysterious and unprovable assumption like "Zorn's Lemma." Which problem am I referring to and what's the assumption?