Equivalence Relations

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A *relation* on a set X is just a subset of $X \times X$

[$X \times X$ is the "cartesian product" $X \times X = \{(x, x') : x, x' \in X\}$. The cartesian product is the direct product in the category of sets.]

Notation: If R is a relation on set X,

$x R x'$ means (x, x') is in the relation R.

An *equivalence relation* is a relation with some extra properties. Relation E on X is an equivalence relation if

(a) $x E x$ for all $x \in X$

(b) $x E y$ implies $y E x$ for all $x, y \in X$

(c) $x E y$ and $y E z$ implies $x E z$ for all $x, y, z \in X$.

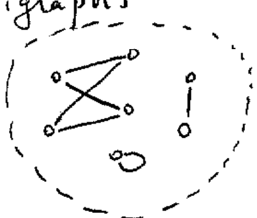
More notation: $[x] \equiv \{x' \in X : x' E x\}$

Such a subset of X is called an "equivalence class".

The set of equivalence classes is ~~called~~ denoted X / E.

examples:

Bigraphs

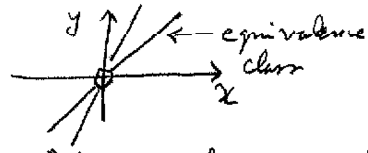
$X =$ 

x "is connected to" y is an equivalence relation.

X/E are the connected components of the Bigraph.

$X = \mathbb{R} \times \mathbb{R}$ i.e. the x-y plane

let $(x, y) \sim (x', y')$ if $(x, y) = (ax', ay')$ for some nonzero $a \in \mathbb{R}$.



$\mathbb{R}^2/\sim$ are lines

$X = \mathbb{R}$

$x \sim x'$ if $x - x' = m 2\pi$ is an equivalence relation.

$\mathbb{R}/\sim$ is isomorphic to the circle

From the category theoretic point of view, X/E is called a "Quotient". Here, X/E is a set and it's not that impressive. However in other categories, the analogous construction will produce quotient groups, quotient vector spaces, quotient topologies etc. In all these situations, there will be a construction like this:

Suppose E is an equivalence relation on X and $X \xrightarrow{f} Y$ "respects the structure of E" in the sense that $xEx' \Rightarrow f(x) = f(x')$. Let $\alpha : x \mapsto [x]$ be the "natural insertion" of $x \in X$ into its equivalence class $[x] \in X/E$. Then, there is a unique γ such that

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "f"'] & X/E \arrow[d, "\gamma"] \\ & Y \end{tikzcd}$$

$\alpha : x \mapsto [x]$ is obviously an epimorphism.

commutes.

<u>Proof.</u> Let $\gamma : \alpha(x) \mapsto f(x)$. We have to check that γ is a function. Since α is epi, we only have to check that $\alpha(x) = \alpha(x') \Rightarrow \gamma(x) = \gamma(x')$. $\alpha(x) = \alpha(x') \Rightarrow [x] = [x'] \Rightarrow xEx' \Rightarrow f(x) = f(x') \Rightarrow \gamma(\alpha(x)) = \gamma(\alpha(x')) \Rightarrow \gamma$ is a function. Since α is epi, γ is unique because if some γ' also makes the diagram commute, then

$$\begin{tikzcd} X \arrow[r, "\alpha"] & X/E \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\gamma'"'] & Y \end{tikzcd}$$ commutes $\Rightarrow \gamma = \gamma'$.