S.Y. March 2000

Let's do some E & M on $\mathbb{R}^3$

From last time, we have

$$\begin{array}{ccccccc} \Lambda^0(\mathbb{R}^3) & \xrightarrow{d} & \Lambda^1(\mathbb{R}^3) & \xrightarrow{d} & \Lambda^2(\mathbb{R}^3) & \xrightarrow{d} & \Lambda^3(\mathbb{R}^3) \\ \cong \mathbb{R} & & \cong \mathbb{R}^3 & & \cong \mathbb{R}^3 & & \cong \mathbb{R} \end{array}$$ de Rham Complex

+ "pullback" co-functor, Hodge star, Stoke's theorem.

From e.g. Jackson, the electric field of a charge q is $\mathbb{E} = q \dfrac{\mathbb{x}}{|\mathbb{x}|^3}$

In our language, this must be a 1-form or a 2-form. We can use the inner product isomorphism

$v \mapsto (v' \mapsto \langle v, v' \rangle)$ to get a 1-form.

$v' \mapsto \langle \mathbb{E}, v' \rangle \in (\mathbb{R}^3)^* = \Lambda^1(\mathbb{R}^3)$

$(x, y, z) \mapsto E_x x + E_y y + E_z z$

$= E_x\, dx + E_y\, dy + E_z\, dz$: a 1-form.

In our case, we have

$$E = \frac{e}{r^2}(x\,dx + y\,dy + z\,dz) \quad \text{on } \mathbb{R}^3 - \{0\}$$

Exercise. Show that $dE = 0$, i.e. E is closed.

Since $\pi_1(\mathbb{R}^3 - \{0\})$ is trivial, E is also exact (see previous notes).

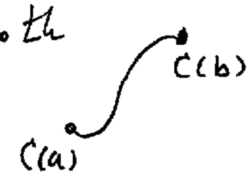
$\Rightarrow E = d\varphi$ for some 0-form φ.

$$\frac{e}{r^2}(x\,dx + y\,dy + z\,dz) = d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz$$

$\varphi = -\frac{e}{r}$ does the trick.

As usual, we can pull back E to any convenient coordinate system.

Suppose we want to integrate E along a smooth curve $I \xrightarrow{c} \mathbb{R}^3$, $I \equiv [a,b]$.



$$\int_c E = \int_I c^* E = \int_I c^* d\varphi = \int_I d(\varphi \circ c) = \varphi(c(b)) - \varphi(c(a))$$

How about a monopole?

$$\mathbb{B} = \frac{g\,\mathbb{X}}{|\mathbb{X}|^3}$$

The difference is that $\mathbb{B}$ is actually a 2-form which we get by applying the Hodge star

$$B = *(B_x\,dx + B_y\,dy + B_z\,dz)$$

$$= B_x\,dy\wedge dz + B_y\,dz\wedge dx + B_z\,dx\wedge dy$$

$$= \frac{g}{r^2}(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy) \text{ on } \mathbb{R}^3 - \{0\}$$

Exercise: Show that $dB = 0$ also.

Is B exact also? $B = dA$?

Poincaré: Yes on a contractible subset of $\mathbb{R}^3$.

$\Rightarrow$ $B = dA$ on any contractible subset of $\mathbb{R}^3 - \{0\}$, but all of $\mathbb{R}^3 - \{0\}$ is not contractible even though $\pi_1(\mathbb{R}^3 - \{0\})$ is trivial.

There is analogous results from the <u>second</u> homotopy group $\pi_2(\mathbb{R} - \{0\})$. We haven't covered this, but informally, $\pi_2(X)$ is trivial if every 2-sphere can be interpolated to a point.

Theorem. A closed k-form on X with $\pi_2(X)$ trivial is exact.

$\mathbb{R}^3 - \{0\}$ does not have trivial π_2, but if we exclude a line, then every sphere on $\mathbb{R}^3$ - line is contractible

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$\Rightarrow$ $B = dA$ on $\mathbb{R}^3$ - line.

This is the Dirac string! You can choose two strings and define e.g.

$$A_+ = \frac{g}{r}\frac{1}{z+r}(x\,dy - y\,dx) \quad \text{on} \quad \mathbb{R}^3 - Z_-$$

$$A_- = \frac{g}{r}\frac{1}{z-r}(x\,dy - y\,dx) \quad \text{on} \quad \mathbb{R}^3 - Z_+$$

What does Stoke's theorem tell us?

$$\int_M dA = \int_{\partial M} i^* A \qquad \begin{array}{l} M \text{ any 2-manifold} \\ \text{with 1-dimensional boundary } \partial M. \end{array}$$

Rotations, SU(2) etc.

The key to a nice understanding of this comes from the quaternions $\mathcal{H}$.

$\mathcal{H}$ is simply $\mathbb{R}^4$ with a norm and a bilinear associative product (only $\mathbb{R}, \mathbb{C}, \mathcal{H}$ have these properties).

The traditional basis for $\mathcal{H}$ is $\{1, i, j, k\} \to \mathcal{H}$. Since the product is bilinear, we can use

$$\begin{tikzcd} \mathcal{H} \times \mathcal{H} \arrow[r, "\otimes"] \arrow[dr, "\text{product}"'] & \mathcal{H} \otimes \mathcal{H} \arrow[d, "\gamma"] \\ & \mathcal{H} \end{tikzcd}$$

to notice that we only need to define the product on $\{1, i, j, k\} \times \{1, i, j, k\}$

Let $1^2 = 1$, $i^2 = j^2 = k^2 = -1$

$ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$

note that the product is not commutative.

Quaternions also have a nice representation as $(r, \vec{A})$ where $r \in \mathbb{R}$ and $\vec{A}$ is a three vector. Then the product is

$$(r, \vec{A}) \cdot (r', \vec{A}') = (rr' - \vec{A} \cdot \vec{A}', r\vec{A} + r'\vec{A}' + \vec{A} \times \vec{A}')$$

The conjugate of a quaternion $q = a \cdot 1 + b \cdot i + c \cdot j + d \cdot k$ is $\bar{q} = a \cdot 1 - b \cdot i - c \cdot j - d \cdot k$. You can easily check that

$q\bar{q} = a^2 + b^2 + c^2 + d^2 = |q|^{1/2}$ where $|q| \equiv (q\bar{q})^{1/2}$ is a norm.

Also, $\overline{q q'} = \bar{q}' \bar{q}$, so

$$|q q'|^2 = q q' \bar{q}' \bar{q} = |q|^2 |q'|^2 \Rightarrow |q q'| = |q||q'|.$$

Notice that any nonzero quaternion q has an inverse $\frac{1}{|q|^2} \bar{q}$. The set of *unit* quaternions

$$q = 1a + ib + jc + kd \quad \text{s.t. } a^2 + b^2 + c^2 + d^2 = 1$$

is thus a group which we can identify with S^3 the three sphere in $\mathbb{R}^4$ (S^1 and S^3 are groups, but S^2 isn't!).

q above is called *pure* if $a = 0$.

Whenever you have a group like $S^3 \subset \mathcal{H}$, it's interesting to look at left + right action. For $q \in S^3$,

$$x \mapsto q x \qquad\qquad x \mapsto x q$$

$$x \mapsto q x q^{-1} \qquad \text{also "conjugation"}$$

These are linear maps $\mathcal{H} \to \mathcal{H}$ preserving the norm (e.g. $|q x - q x'| = |q(x - x')| = |q| |(x - x')| = |x - x'|$).

They are orientation preserving because S^3 is connected (Why?).

Notice that pure quaternions are $\mathbb{R}^3$ and pure unit quaternions are the 2-sphere S^2 in $\mathbb{R}^3$. Conjugation gives us a map

$$q \mapsto (p \mapsto q p q^{-1}) \quad \text{from}$$

$$S^3 \longrightarrow (S^2 \to S^2)$$

norm + orientation preserving $\Rightarrow \in SO(3)$.

Exercise: Let S^2 be the sphere of pure quaternions with unit norm. Show that conjugation by a unit quaternion $(r, \vec{A})$ rotates elements of S^2 around the axis $\vec{A}$ by $2\tan^{-1}(|\vec{A}|/r)$.

Exercise: Show that the kernel of $\varphi: q \mapsto (p \mapsto q p q^{-1})$ is the subgroup $\{\pm 1\}$ of $\mathcal{H}$.

Thus, we have

$$\begin{tikzcd} S^3 \arrow[r, "\alpha"] \arrow[dr, "\varphi"'] & S^3/\{\pm 1\} \arrow[d, "\gamma"] \\ & SO(3) \end{tikzcd}$$

For a unique mono γ. From the exercise above, φ is also epi $\Rightarrow$ $S^3/\{\pm 1\} \cong SO(3)$.

We can also identify $\mathcal{H} \cong \mathbb{C} \times \mathbb{C}$. Here multiplication

$$(a,b)(a',b') = (a a' - \bar{b} b', b a' + \bar{a} b')$$

looks like complex multiplication with a few "$-$" thrown in.

Exercise: Show that this is the quaternion multiplication.

Now notice that the map $(a,b) \mapsto \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ from unit quaternions S^3 to $SU(2)$ matrices is an isomorphism.

We have

$$S^3 \cong SU(2) \qquad S^3/\{\pm 1\} \cong SU(2)/\{\pm 1\} \cong SO(3)$$

epi α above is, in fact, a double covering as defined before.

I keep saying that matrix groups like $SO(3)$ are actually manifolds $\Rightarrow$ topological spaces. How is this done?

$$\begin{pmatrix} a^{11} & a^{12} & \cdots & a^{1n} \\ a^{21} & & & \vdots \\ \vdots & & & \\ a^{m1} & - & - & a^{mn} \end{pmatrix} \mapsto (a^{11}, a^{12}, \cdots a^{1m}, a^{21}, \cdots, a^{mn})$$

$$\in \mathbb{R}^{mn}$$

Simply use the topology induced by this linear map.

Since det is a polynomial in the matrix elements, it is continuous. This means that

$$n \times n \text{ matrices } M^n \cong \mathrm{End}(\mathbb{R}^n) \xrightarrow{\det} \mathbb{R}$$

$\mathrm{Aut}(\mathbb{R}^n)$ is the pullback of the open set $\mathbb{R} - \{0\}$

$\Rightarrow$ $\mathrm{Aut}(\mathbb{R}^n)$ is an open subset of $\mathrm{End}(\mathbb{R}^n)$,

$GL(n)$ is an open subset of M^n.

As in most categories, it is much easier to obtain objects by indirect methods rather than by applying the definitions directly. This is especially true for manifolds.

For example, we would like to know that S^2 is a manifold because it is the pull back of

$$r : (x,y,z) \mapsto x^2+y^2+z^2$$

$$S^2 = r^{-1}[1].$$

This is OK provided the differential of the map is epimorphic at each point in the pull back.

Def. $M \subset \mathbb{R}^{n+k}$ is an <u>n-dimensional submanifold</u> of $\mathbb{R}^{n+k}$ if, for any $x \in M$, there exists an open $\mathcal{U} \ni x$ $\mathcal{U} \subset \mathbb{R}^{n+k}$ such that $\varphi : \mathcal{U} \to \mathbb{R}^k$, $d\varphi_x$ is epi on $x \in M$, $\mathcal{U} \cap M = \varphi^{-1}[0]$.

Example: $(k=1)$

$$S^n = \{ x \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 - 1 = 0 \}$$

is a submanifold because $\varphi : x \mapsto x_1^2 + x_2^2 + \cdots + x_{n+1}^2 - 1$ has epi differential.

example:

$$SO(n) = \{ A \in M^n : \det(A) > 0 \text{ and } {}^tAA = 1 \}$$

is an $\frac{n(n-1)}{2}$ dimensional submanifold of $M^n \cong \mathbb{R}^{n^2}$.

To see this, let $GL_n^+ \supset SO(n)$ be the group of $n \times n$ matrices with positive determinant.

Let $f : GL_n^+ \to Sym(n)$ be given by

$$f : A \mapsto {}^tAA - 1$$

Then $SO(n) = f^{-1}[0]$ and

$df_A : H \mapsto {}^tAH + {}^tHA$ is epi for $A \in SO(n)$

[For symmetric S, $df_A\left(\frac{AS}{2}\right) = S$].

A submanifold, as we have defined it, is also a manifold.