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Topological Vector Spaces

Let's see if we can approximate a continuous function

$$X \xrightarrow{f} Y \quad \text{with a linear function}$$

$$X \xrightarrow{L} Y \quad L \in \mathrm{Lin}(X;Y).$$

For this to be meaningful, X, Y should have linear structure as well as a topology. We'll assume that X and Y are real, finite-dimensional "topological vector spaces." This is just a Hausdorff space where $(x,x') \mapsto x+x'$, $(a,x) \mapsto ax$ and $x \mapsto -x$ are continuous. Geroch proves that any such space is isomorphic to the euclidean topology on $\mathbb{R}^n$ given by the metric

$$d : (x, x') \longmapsto |x - x'|$$

where $|\,| : X \to \mathbb{R}$ is a norm i.e.

$$|ax| = |a||x| \qquad a \in R$$

$$|x + x'| \leq |x| + |x'|$$

$$|x| \geq 0 \qquad |x| = 0 \Rightarrow x = 0.$$

Exercise: Prove that d, as defined, is a metric.

We can now use the norm to measure how close a linear map is to a continuous map.

Aside: Is $\mathbb{R}^n$ boring? How about this: Are all manifolds on topological space $R^n$ isomorphic?
Answer: Yes, except for $n=4$ when there are an infinite number of solutions. Donaldson, 1986 Fields Medal.

$$X \xrightarrow{f} Y \qquad f \in \mathrm{Mor}(X, Y)$$

$$X \xrightarrow{L} Y \qquad L \in \mathrm{Lin}(X; Y)$$

For a fixed $x \in X$, let $R$ be defined by

$$f(x+h) = f(x) + L(h) + R(X, h)$$

If $h \mapsto \begin{cases} |R(X,h)| / |h| & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$ is continuous

then $f$ is said to be <u>differentiable at $x$</u>. In this case, $L$ is called the <u>differential</u> of $f$ at $x$, often denoted $df_x \in \mathrm{Lin}(X;Y)$.

The differentials have various nice properties, e.g.

<u>Theorem</u>: Differentials are unique.

Proof. Suppose that $L$ and $L'$ are both differentials of $f$ at $x$ and $L$ and $L'$ ~~diffles~~ differ at some nonzero $h \in X$.

Then $\dfrac{L(h) - L'(h)}{|h|} = \dfrac{R'(x,h)}{|h|} - \dfrac{R(X,h)}{|h|}$ if $h \neq 0$

$$\Rightarrow \left| (L - L')(h/|h|) \right| = \left| \frac{R'(x,h)}{|h|} - \frac{R(x,h)}{|h|} \right| \leq \frac{|R'(x,h)|}{|h|} + \frac{|R(x,h)|}{|h|}.$$

Since the quantity on the left is constant on the line $a \cdot h$ $a \in \mathbb{R}^+$, any neighborhood of $O$ in $X$ contains such a point (why?) and so $h \mapsto \begin{cases} \dfrac{|R'(x,h)|}{|h|} + \dfrac{|R(x,h)|}{|h|} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$ is not continuous $\Rightarrow$ $L$ or $L'$ are not differentials of $f$ at $x \Rightarrow L = L'$.

exercise: Fill in the details ~~de~~ of this proof.

Besides uniqueness, the most important fact about differentials is

<u>Theorem (The chain rule)</u>: If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are differentiable, then $X \xrightarrow{g\circ f} Z$ is differentiable and, at any $x \in X$, $d(g\circ f)_x = dg_{f(x)} \circ df_x$.

Proof. Given $x \in X$, let $f(x+h) = f(x) + df_x(h) + R_f(x,h)$.
Then $g(f(x+h)) = g(f(x)) + dg_{f(x)} \circ df_x(h) + R(x,h)$

where $R(x,h) = dg_{f(x)}(R_f(x,h)) + R_g(f(x), df_x(h) + R_f(x,h))$.

Since $h \mapsto \begin{matrix} |R(x,h)|/|h| & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{matrix}$ is continuous, we're done.

exercise: Fill in the details of the proof.

Notice that these two results give us a functor

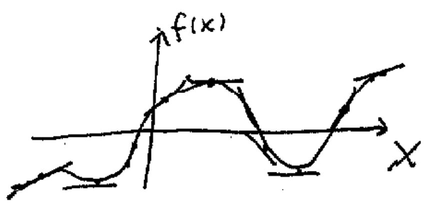
$$(X, x) \xrightarrow{f} (Y, y) \qquad \text{Pointed topological spaces}$$

$$T(X, x) \xrightarrow{Tf} T(Y, y) \qquad \text{Real vector spaces, } Tf \equiv df_x$$

$T$ is called the "Tangent space" functor.

Notice how the functorial condition $T(g\circ f) = Tg \circ Tf$ is exactly the chain rule.

It's often useful to look at the differential as a "section" of a fiber bundle.




Let $TX \equiv \{(x,v) : x \in X,\ v \in T(X,x)\}$

This is the "tangent bundle" with projection $TX \xrightarrow{\pi} X$, $\pi : (x,v) \mapsto x$.

Then, the differential of $f$ is a particular section of $TX$, i.e. a $TX \xleftarrow{d} X$ such that $\pi \circ d = i_X$ i.e. a choice of $v \in T(X,x)$ for every $x \in X$.

In general the category of bundles over $X$ is

Objects: $E \xrightarrow{\pi} X$, $F \xrightarrow{\nu} X$, etc.

Morphisms: $(E,\pi) \xrightarrow{\varphi} (F,\nu)$ s.t.
$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$
commutes.

Bundles are a convenient way to define, e.g., vector fields. A vector field on manifold $M$ is a section of $TM \xrightarrow{\pi} M$ i.e. for each $p \in M$, a vector

$$v(p) = a_1(p)\,\epsilon_1^p + a_2(p)\,\epsilon_2^p + \dots + a_m(p)\,\epsilon_m^p$$

where $\{\epsilon_1^p, \epsilon_2^p, \dots, \epsilon_m^p\}$ are a basis of $T(M,p) \equiv$ "$T_p(M)$".

# Differential Forms

Let's consider differentiable functions $\mathcal{U} \xrightarrow{f} \mathbb{R}$ where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$. The simplest such functions are projections

$$\text{"}x_i\text{"} : (a_1, a_2, \dots, a_i, \dots, a_n) \mapsto a_i$$

Since $x_i(x+h) = x_i(x) + x_i(h) + 0$, $x_i$ is differentiable with $dx_i : h \mapsto h_i$. Notice that $dx_i \in \mathrm{Lin}(\mathbb{R}^n; \mathbb{R}) = (\mathbb{R}^n)^*$, and $\{dx_1, dx_2, \dots, dx_n\}$ are a basis for $(\mathbb{R}^n)^*$.

Thus, in general,

$$df_x(h) = \sum_{j=1}^{n} a_j(x)\, dx_j(h)$$

The $a_j(x)$ are called partial derivatives, usually written $a_j(x) = \dfrac{\partial f}{\partial x_j}(x)$.

exercise: Consider $f(x + \lambda e_j)$ and show that $a_j(x)$ is the partial derivative of $f$ defined in the usual way.

Given any vector space $V$, (e.g. $(\mathbb{R}^n)^*$), we have already defined $V \wedge V$, $V \wedge V \wedge V$, etc., meaning that

$$dx \wedge dy + dy \wedge dz \in \Lambda^3 V \quad \text{etc. are defined.}$$

We would like to extend calculus to these geometric objects as well.

If we define a "k-uple" $I = (i_1, i_2, \cdots, i_k)$ to be an increasing list of $k$ integers between $1$ and $n$, then

$$dx_I \equiv dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

is a basis of $\Lambda^k(\mathbb{R}^n)^*$. Thus, the general $k$-form can be written

$$\omega = \sum_I{}' a_I \, dx_I \qquad a_I \in \mathbb{R}$$

note that a $0$-form is a real constant. We can extend this to $U \subset \mathbb{R}^n$ by letting

$$\omega(x) = \sum_I{}' a_I(x) \, dx_I$$

where $a_I(x)$ are smooth functions from $U$ to $\mathbb{R}$. In this context, $\omega$ is called a ~~an~~ differentiable $k$-form. Notice that differentiable $0$-forms are simply real valued functions on $U$. This means that we already have

$$\Lambda^0(\mathbb{R}^n)^* \xrightarrow{d} \Lambda^1(\mathbb{R}^n)^*$$

i.e. the differential. We can extend this to $\Lambda^k(\mathbb{R}^n)^* \xrightarrow{d} \Lambda^{k+1}(\mathbb{R}^n)^*$ by

$$d\omega \equiv \sum_I{}' da_I \wedge dx_I$$

$d$ is called the "exterior derivative".

The exterior derivative has these properties

a) If $f \in \Lambda^0(\mathbb{R}^n)^*$, $d(f)$ is the differential

b) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$

c) $d(d\omega) = 0$

d) $d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi$ where $\omega$ is a k-form and $\varphi$ is an s-form.

Exercise. Prove a), b), d) above.

To prove c), let $f : \mathbb{R}^n \to \mathbb{R}$ be a 0-form. Then

$$d(df) = d\Big( \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \Big) = \sum_{j=1}^{n} d\Big(\frac{\partial f}{\partial x_j}\Big) \wedge dx_j$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k} dx_k \wedge dx_j = 0 \quad \text{since} \quad \frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j}$$

for smooth $f$. Using d), this is true for any k-form.

If $d\omega = 0$, $\omega$ is called <u>closed</u>, i.e. $\omega \in \operatorname{Ker} d$.

If $\omega = d\varphi$ for some form $\varphi$, $\omega$ is called <u>exact</u>, i.e. $\omega \in \operatorname{Im} d$.

Since $d^2 = 0$, every exact form is closed.

Theorem (Poincaré): If $\mathcal{U} \subset \mathbb{R}^n$ is contractible, then a k-form on $\mathcal{U}$ is closed iff it is exact.

Example: $\mathbb{R}^3$

For $f \in \Lambda^0(\mathbb{R}^3)^*$,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

For $f_1\,dx + f_2\,dy + f_3\,dz \in \Lambda^1(\mathbb{R}^3)^*$

$$d(f_1\,dx + f_2\,dy + f_3\,dz) =$$

$$\left(\frac{\partial f_1}{\partial y} - \frac{\partial f_2}{\partial x}\right) dy \wedge dx - \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dx \wedge dz + \left(\frac{\partial f_2}{\partial z} - \frac{\partial f_3}{\partial y}\right) dy \wedge dz$$

For $\omega = f_1\,dy \wedge dz + f_2\,dz \wedge dx + f_3\,dx \wedge dy \in \Lambda^2(\mathbb{R}^3)^*$

$$d\omega = \frac{\partial f_1}{\partial x}\,dx \wedge dy \wedge dz + \frac{\partial f_2}{\partial y}\,dy \wedge dz \wedge dx + \frac{\partial f_3}{\partial z}\,dz \wedge dx \wedge dy$$

$$= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx \wedge dy \wedge dz$$

$d$ (0-form) $\cong$ Gradient

$d$ (1-form) $\cong$ Curl

$d$ (2-form) $\cong$ divergence

Unlike tensor fields, differential forms can be pulled back across smooth maps.

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \qquad f \text{ smooth}$$

If we have a k-form $\omega$ on $\mathbb{R}^m$, we can define

$$\omega = \sum_I{}' \, a_I \, dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_k}$$

$$f^*\omega = \sum_I{}' (a_I \circ f) \; d(y_{i_1} \circ f) \wedge d(y_{i_2} \circ f) \wedge \cdots \wedge d(y_{i_k} \circ f)$$

Pullbacks have these properties

(a) $f^*(\omega + \omega') = f^*\omega + f^*\omega'$

(b) $f^*(g\,\omega) = f^*(g)\, f^*(\omega)$

(c) $f^*(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k) = f^*(\varphi_1) \wedge \cdots \wedge f^*(\varphi_k)$

for 1-forms $\varphi_k$.

They also interact nicely with the exterior derivative

$$d(f^*\omega) = f^*(d\omega).$$

~~For isomorphism $f^*$, $f^*$ corresponds to~~

Example. Let $\mathcal{U} = \{(r,\theta) : r > 0,\ 0 < \theta < 2\pi\} \subset \mathbb{R}^2$

$\varphi : \mathcal{U} \to \mathbb{R}^2$ be $\varphi : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$.

Then the pullback of the volume form $dx \wedge dy$ is given by

$$\varphi^*(dx \wedge dy) = d(x\circ\varphi) \wedge d(y\circ\varphi) = r\, dr \wedge d\theta$$

Exercise: Fill in the steps

Exercise: Let $\omega = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$ be a 1-form in $\mathbb{R}^2 - \{0\}$. Show that $\varphi^*\omega = d\theta$.

Exercise: Let $f : (r,\theta,\varphi) \mapsto (r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi,\ r\cos\theta)$. Show that $f^*(dx \wedge dy \wedge dz) = r^2 \sin\theta\, dr \wedge d\theta \wedge d\varphi$.

Notice that the pullback of an isomorphism is "change of variables".

The sequence

$$\Lambda^0(\mathbb{R}^n)^* \xrightarrow{d} \Lambda^1(\mathbb{R}^n)^* \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\mathbb{R}^n)^*$$

is called "the de Rham complex". $\Lambda^*(\mathbb{R}^n)^* \equiv \bigoplus_K \Lambda^K(\mathbb{R}^n)^*$

The direct sum of vector spaces $H^K \equiv \mathrm{Ker}\, d / \mathrm{Im}\, d$

$\bigoplus_K H^K$ is called "the de Rham cohomology."

de Rham showed that this is a topological invariant.

More generally, a direct sum $\bigoplus_K V^K$ of vector spaces with

$$V^0 \xrightarrow{d} V^1 \xrightarrow{d} V^2 \xrightarrow{d} \cdots$$

$d^2 = 0$, is called a <u>differential complex</u>. $\bigoplus_K \mathrm{Ker}\, d / \mathrm{Im}\, d$ is called the cohomology of $V$. A morphism between differential complices $A$ and $B$ is defined to be a sequence of morphisms $f : A^K \to B^K$ s.t.

$$\begin{tikzcd} A^0 \arrow[r, "d"] \arrow[d, "f"'] & A^1 \arrow[r, "d"] \arrow[d, "f"] & A^2 \arrow[r, "d"] \arrow[d, "f"] & \cdots \arrow[d, "f"] \\ B^0 \arrow[r, "d"] & B^1 \arrow[r, "d"] & B^2 \arrow[r, "d"] & \cdots \end{tikzcd}$$

commutes. In this language, $\Lambda^*$ is a cofunctor from the category of Euclidean spaces $\mathbb{R}^n$ to the category of differential complexes

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \qquad \Lambda^* \mathbb{R}^n \xleftarrow{f^*} \Lambda^* \mathbb{R}^m$$

where $f^*$ is the pull back.

# Hodge Star

An <u>inner product</u> $\langle\,\rangle : V \times V \to \mathbb{R}$ on a real finite dimensional vector space $V$ is a bilinear symmetric product which is "nondegenerate" meaning that

$$\langle v, x \rangle = 0 \text{ for all } x \in V \Rightarrow v = 0.$$

<u>Lemma</u>. For $f \in V^*$, $f = (v \mapsto \langle v_f, v \rangle)$ for some unique $v_f \in V$.

Proof. $\Psi : x \mapsto (v \mapsto \langle x, v \rangle)$ from $V$ to $V^*$ is one-to-one (using the nondegenerate property) $\Rightarrow \Psi$ is an isomorphism.

Let $\dim(V) = n$, $\overbrace{V \wedge V \wedge \cdots \wedge V}^{k} = \Lambda^k$. Fix $\lambda \in \Lambda^k$.

Then the mapping

$$\mu \mapsto \lambda \wedge \mu \quad \text{from } \Lambda^{n-k} \text{ to } \Lambda^n \cong \mathbb{R} \text{ is}$$

$$\lambda \wedge \mu = \langle v_\lambda, \mu \rangle \sigma \quad \text{for some fixed basis vector } \sigma \in \Lambda^n$$

for some special unique $v_\lambda \in \Lambda^{n-k}$. This $v_\lambda$ is denoted $*\lambda$, the "Hodge Star" of $\lambda$. Notice that $*\lambda$ depends both on the inner product and on the choice of basis $\sigma \in \Lambda^n$.

For example, let $n=3$, $\epsilon_x, \epsilon_y, \epsilon_z$ be an orthonormal basis in $\mathbb{R}^3$, $\langle\rangle$ the standard euclidean inner product, let $\sigma = \epsilon_x \wedge \epsilon_y \wedge \epsilon_z$.

Then $\epsilon_x \wedge \epsilon_y \wedge \left( \sum_i a_i \epsilon_i \right) = \langle *(\epsilon_x \wedge \epsilon_y), \sum_i a_i \epsilon_i \rangle \, \epsilon_x \wedge \epsilon_y \wedge \epsilon_z$

is solved by $*(\epsilon_x \wedge \epsilon_y) = \epsilon_z$.

Similarly,

$$*\epsilon_x = \epsilon_y \wedge \epsilon_z \qquad *(\epsilon_x \wedge \epsilon_y) = \epsilon_z$$
$$*\epsilon_y = \epsilon_z \wedge \epsilon_x \qquad *(\epsilon_y \wedge \epsilon_z) = \epsilon_x \qquad *(\epsilon_x \wedge \epsilon_y \wedge \epsilon_z) = 1.$$
$$*\epsilon_z = \epsilon_x \wedge \epsilon_y \qquad *(\epsilon_z \wedge \epsilon_x) = \epsilon_y$$

Exercise. Show that $\vec{A} \times \vec{B} = *(A \wedge B)$.

The Hodge star has this property: $\alpha \wedge * \beta = \beta \wedge * \alpha$

Exercise: For $f : \mathbb{R}^n \to \mathbb{R}$, show that $d*(df) = \Delta f \mu$ where $\mu$ is the volume form and $\Delta$ is the laplacian.

Example. With $F_{\mu\nu}$ the EM field strength, $F = \frac{1}{2} F_{\mu\nu} \, dx_\mu \wedge dx_\nu$ is the electromagnetic field. Maxwell's equations are

$$dF = 0$$
$$*d*F = *j$$

$\rightarrow$ F is closed. If it is defined on a contractable subset of $\mathbb{R}^4$, then $F = dA$ by Poincaré's lemma.