University of Pardubice
Faculty of Chemical Technology

Physics

Syllabus
Study material
Auto-evaluation test

Doc. RNDr. Miloš Steinhart, CSc.

UNIVERSITY
OF PARDUBICE
FACULTY
OF CHEMICAL
TECHNOLOGY

Course description

Course abbreviation:	UAFM/C009A	**Page:**	1 / 2
Course name:	Physics		
Academic Year:	2018/2019	**Printed:**	02.07.2018 13:03

Department/Unit /	UAFM / C009A			**Academic Year**	2018/2019
Title	Physics			**Type of completion**	Examination
Accredited/Credits	Yes, 5 Cred.			**Type of completion**	Combined
Number of hours	Lecture 2 [HOD/TYD] Seminar 1 [HOD/TYD]				
Occ/max	Status A	Status B	Status C	**Course credit prior to**	NO
Summer semester	0 / -	0 / -	0 / -	**Counted into average**	YES
Winter semester	0 / -	0 / -	0 / -	**Min. (B+C) students**	not determined
Timetable	Yes			**Repeated registration**	NO
Language of instruction	English			**Semester taught**	Summer semester
Optional course	Yes			**Internship duration**	0
Automat. uzn. záp. před	No				
Periodicita					
Substituted course	None				
Preclusive courses	N/A				
Prerequisite	N/A				
Informally recommended courses	N/A				
Courses depending on this Course	N/A				

Course objectives:

Introduction into basic principles of physics emphasizing Newtonian mechanics, conservation laws, thermal physics, electricity and magnetism, waves, geometrical optics, atomic and nuclear physics.

Requirements on student

Content

Introduction into physics and necessary basic mathematics. SI units.
Kinematics of mass points.
Dynamics of mass points and rigid bodies. Newton's laws.
Conservation laws. Energy, linear and rotational momentum.
Gravitation, intensity, potential energy, potential, conservative fields.
Elasticity and fracture. Stress, strain, Hook's law.
Fluids. Hydrostatics and hydrodynamics of ideal liquids. Newton's fluids.
Oscillations and wave motion. Doppler's effect.
Temperature and heat. Thermal expansion. Heat capacity.
Basic thermodynamics.
Electrostatics. Electric charge and field. Coulomb's law. Gauss 's law.
Capacitance. Electric potential.
Electrokinetics. Electric current and resistance. DC circuits.
Magnetism.
Electromagnetism. Lorentz force, Faraday's law.
Basics of Optics. Reflection and refraction, lenses, optical instruments.
Introduction into modern physics. Black body radiation. Photoelectric effect.
X-Rays.
Introduction into Atomic and nuclear physics.

Prerequisites - other information about course preconditions

Basic level of algebra, trigonometry and calculus

Competences acquired

The subject widens and deepens the secondary school physics knowledge. Prepares students for further studies. Enables them to solve simple and more advanced problems.

Fields of study

Guarantors and lecturers

- **Guarantors:** doc. RNDr. Miloš Steinhart, CSc.
- **Lecturer:** doc. RNDr. Miloš Steinhart, CSc.
- **Seminar lecturer:** RNDr. Petr Janíček, Ph.D., doc. RNDr. Miloš Steinhart, CSc.

Literature

- **Recommended:** Halliday, Resnick, Walker. *Fundamentals of Physics*.
- **Recommended:** Giancoli. *Physics for Scientists and Engineers*.

Teaching methods

Monologic (reading, lecture, briefing)
Dialogic (discussion, interview, brainstorming)

Assessment methods

Oral examination
Written examination

Course is included in study programmes:

Course description

Course abbreviation:	UAFM/CD09	**Page:**	1 / 6
Course name:	Physics		
Academic Year:	2018/2019	**Printed:**	02.07.2018 13:01

Department/Unit /	UAFM / CD09	**Academic Year**	2018/2019
Title	Physics	**Type of completion**	Examination
Accredited/Credits	Yes, 5 Cred.	**Type of completion**	Combined
Number of hours	Lecture 2 [HOD/TYD] Seminar 1 [HOD/TYD]		

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Language of instruction	English	**Semester taught**	Summer semester
Optional course	Yes	**Internship duration**	0
Automat. uzn. záp. před	No		
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Preclusive courses	N/A		
Prerequisite	N/A		
Informally recommended courses	N/A		
Courses depending on this Course	N/A		

Course objectives:

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Oscillations and wave motion. Doppler's effect.
Temperature and heat. Thermal expansion. Heat capacity.
Basic thermodynamics.
Electrostatics. Electric charge and field. Coulomb's law. Gauss 's law.
Capacitance. Electric potential.
Electrokinetics. Electric current and resistance. DC circuits.
Magnetism.
Electromagnetism. Lorentz force, Faraday's law.
Basics of Optics. Reflection and refraction, lenses, optical instruments.
Introduction into modern physics. Black body radiation. Photoelectric effect.
X-Rays.
Introduction into Atomic and nuclear physics.

Prerequisites - other information about course preconditions

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The subject widens and deepens the secondary school physics knowledge. Prepares students for further studies. Enables them to solve simple and more advanced problems.

Fields of study

Guarantors and lecturers

- **Guarantors:** doc. RNDr. Miloš Steinhart, CSc.
- **Lecturer:** doc. RNDr. Miloš Steinhart, CSc.
- **Seminar lecturer:** RNDr. Petr Janíček, Ph.D., doc. RNDr. Miloš Steinhart, CSc.

Literature

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Teaching methods

Monologic (reading, lecture, briefing)
Dialogic (discussion, interview, brainstorming)

Assessment methods

Oral examination
Written examination

Course is included in study programmes:

Study Programme	Type of	Form of	Branch	Stage	St. plan v.	Year	Block	Status	R.year	R.
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2015	2018	volitelné předměty	C	2	LS
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2013	2018	volitelné předměty	C	3	LS
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2014	2018	volitelné předměty	C	3	LS
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2015	2018	volitelné předměty	C	3	LS
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2016	2018	volitelné předměty	C	3	LS
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2017	2018	volitelné předměty	C	3	LS
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2013	2018	volitelné předměty	C	2	LS
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2018	2018	volitelné předměty	C	2	LS
Farmacochemistry and Medicinal	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2014	2018	volitelné předměty	C	2	LS

Study Programme	Type of	Form of	Branch	Stage	St. plan v.	Year	Block	Status	R.year	R.
Materials										
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2018	2018	volitelné předměty	C	3	LS
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2016	2018	volitelné předměty	C	2	LS
Farmacochemistry and Medicinal Materials	Bachelor	Full-time	Farmacochemistry and Medicinal Materials	1	2017	2018	volitelné předměty	C	2	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2017	2018	volitelné předměty	C	2	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2016	2018	volitelné předměty	C	2	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2018	2018	volitelné předměty	C	3	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2013	2018	volitelné předměty	C	3	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2014	2018	volitelné předměty	C	3	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2015	2018	volitelné předměty	C	3	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2016	2018	volitelné předměty	C	3	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2015	2018	volitelné předměty	C	2	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2014	2018	volitelné předměty	C	2	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2013	2018	volitelné předměty	C	2	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2018	2018	volitelné předměty	C	2	LS
Graphic Arts and Printing Technology	Bachelor	Full-time	Graphic Arts and Printing Technology	1	2017	2018	volitelné předměty	C	3	LS
Chemical and Process Engineering	Bachelor	Full-time	Economy and Management of Chemical and Food Industry	1	2018	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Economy and Management of Chemical and Food Industry	1	2016	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Economy and Management of Chemical and Food Industry	1	2017	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Economy and Management of Chemical and Food Industry	1	2014	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Economy and Management of Chemical and Food Industry	1	2015	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Economy and Management of Chemical and Food Industry	1	2013	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2013	2018	volitelné předměty	C		LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2014	2018	volitelné předměty	C		LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2015	2018	volitelné předměty	C		LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2016	2018	volitelné předměty	C		LS

Study Programme	Type of	Form of	Branch	Stage	St. plan v.	Year	Block	Status	R.year	R.
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2017	2018	volitelné předměty	C		LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2018	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2013	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2014	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2015	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2018	2018	volitelné předměty	C		LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2017	2018	volitelné předměty	C	2	LS
Chemical and Process Engineering	Bachelor	Full-time	Environment Protection	1	2016	2018	volitelné předměty	C	2	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2013	2018	volitelné předměty	C	2	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2017	2018	volitelné předměty	C	2	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2016	2018	volitelné předměty	C	2	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2015	2018	volitelné předměty	C	2	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2014	2018	volitelné předměty	C	2	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2018	2018	volitelné předměty	C	2	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2017	2018	volitelné předměty	C	3	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2016	2018	volitelné předměty	C	3	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2015	2018	volitelné předměty	C	3	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2014	2018	volitelné předměty	C	3	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2013	2018	volitelné předměty	C	3	LS
Chemistry and Technical Chemistry	Bachelor	Full-time	Chemistry and Technical Chemistry	1	2018	2018	volitelné předměty	C	3	LS
Chemistry and Technology of Foodstuffs	Bachelor	Full-time	Evaluation and Analysis of Foodstuffs	1	2017	2018	volitelné předměty	C		LS
Chemistry and Technology of Foodstuffs	Bachelor	Full-time	Evaluation and Analysis of Foodstuffs	1	2016	2018	volitelné předměty	C		LS
Chemistry and Technology of Foodstuffs	Bachelor	Full-time	Evaluation and Analysis of Foodstuffs	1	2015	2018	volitelné předměty	C		LS
Chemistry and Technology of Foodstuffs	Bachelor	Full-time	Evaluation and Analysis of Foodstuffs	1	2018	2018	volitelné předměty	C		LS
Chemistry and Technology of Foodstuffs	Bachelor	Full-time	Evaluation and Analysis of Foodstuffs	1	2013	2018	volitelné předměty	C		LS
Chemistry and Technology of Foodstuffs	Bachelor	Full-time	Evaluation and Analysis of Foodstuffs	1	2014	2018	volitelné předměty	C		LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Inorganic Materials	1	2018	2018	volitelné předměty	C	2	LS

Study Programme	Type of	Form of	Branch	Stage	St. plan v.	Year	Block	Status	R.year	R.
Inorganic and Polymeric Materials	Bachelor	Full-time	Inorganic Materials	1	2015	2018	volitelné předměty	C	2	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Inorganic Materials	1	2017	2018	volitelné předměty	C	2	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Inorganic Materials	1	2016	2018	volitelné předměty	C	2	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Inorganic Materials	1	2018	2018	volitelné předměty	C	3	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Inorganic Materials	1	2015	2018	volitelné předměty	C	3	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Inorganic Materials	1	2016	2018	volitelné předměty	C	3	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Inorganic Materials	1	2017	2018	volitelné předměty	C	3	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Polymeric Materials and Composites	1	2014	2018	volitelné předměty	C	3	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Polymeric Materials and Composites	1	2015	2018	volitelné předměty	C	3	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Polymeric Materials and Composites	1	2016	2018	volitelné předměty	C	3	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Polymeric Materials and Composites	1	2014	2018	volitelné předměty	C	2	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Polymeric Materials and Composites	1	2016	2018	volitelné předměty	C	2	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Polymeric Materials and Composites	1	2017	2018	volitelné předměty	C	3	LS
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Inorganic and Polymeric Materials	Bachelor	Full-time	Polymeric Materials and Composites	1	2018	2018	volitelné předměty	C	3	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Polymeric Materials and Composites	1	2015	2018	volitelné předměty	C	2	LS
Inorganic and Polymeric Materials	Bachelor	Full-time	Polymeric Materials and Composites	1	2018	2018	volitelné předměty	C	2	LS
Surface Protection of Building and Construction Materials	Bachelor	Full-time	Surface Protection of Building and Construction Materials	1	2016	2018	volitelné předměty	C	2	LS
Surface Protection of Building and Construction Materials	Bachelor	Full-time	Surface Protection of Building and Construction Materials	1	2015	2018	volitelné předměty	C	2	LS
Surface Protection of Building and Construction Materials	Bachelor	Full-time	Surface Protection of Building and Construction Materials	1	2018	2018	volitelné předměty	C	2	LS
Surface Protection of Building and Construction Materials	Bachelor	Full-time	Surface Protection of Building and Construction Materials	1	2018	2018	volitelné předměty	C	3	LS
Surface Protection of Building and Construction Materials	Bachelor	Full-time	Surface Protection of Building and Construction Materials	1	2015	2018	volitelné předměty	C	3	LS
Surface Protection of Building and Construction Materials	Bachelor	Full-time	Surface Protection of Building and Construction Materials	1	2016	2018	volitelné předměty	C	3	LS
Surface Protection of Building and Construction	Bachelor	Full-time	Surface Protection of Building and Construction Materials	1	2017	2018	volitelné předměty	C	2	LS

Study Programme	Type of	Form of	Branch	Stage	St. plan v.	Year	Block	Status	R.year	R.
Materials										
Surface Protection of Building and Construction Materials	Bachelor	Full-time	Surface Protection of Building and Construction Materials	1	2017	2018	volitelné předměty	C	3	LS

Physics

1

Introduction into Physics

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Main Topics

1. Introduction
2. Coordinate system
3. Goniometric functions
4. Scalars and vectors, unit vector
5. Basic vector operations
6. Calculus: Derivatives and integrals

Introduction into Physics

- Physics is the most basic science that deals with the structure and behavior of matter (=all around us) from microscopic to macroscopic dimensions.
- Richard Feynman said 'Physics is the way of thinking' : The Nature plays chess, we are watching it and trying to reveal the rules of the game. We can see directly how various stones move but the reason why they move a certain way in a certain situation is a higher level of knowledge.

The Main SI Units

• meter	m	– length
• kilogram	kg	– mass
• second	s	– time
• ampere	A	– electric current
• kelvin	K	– temperature
• mol	mol	– amount of substance
• candela	cd	– illumination

The Main SI Units

The 26th assembly of the general conference for units in Versailles 19. 11. 2018 agreed on redefinition of the main units by fixing the values of 7 physical constants:

- Frequency of transition ^{133}Cs: $f = 9\ 192\ 631\ 770\ Hz$
- Speed of light in vacuum: $c = 299\ 792\ 458\ m/s$
- Planck's constant: $h = 6.626\ 070\ 15.10^{-34}\ J.s$
- Elementary charge: $q_e = 1.602\ 176\ 634.10^{-19}\ C$
- Boltzman's constant: $k_B = 1.380\ 649.10^{-23}\ J/K$
- Avogadro's constant: $N_A = 6.022\ 140\ 76.10^{23}$
- Illumination effectivity of monochromatic irradiation of 540 THz: $K_{cd} = 683\ lm/W$

Length - the meter [m]

- Originally: 10^{-7} part of a quadrant of the Earth. Due to apparent inconvenience of this definition an etalon has been introduced - international meter. Its fundamental advantage over then existing etalons was that it was defined in a reproducible way.
- Recently: the meter is defined on the basis of the speed of the light in vacuum:

$$c = 299\ 792\ 458 \pm 1\ \mathrm{ms}^{-1}$$

Mass - the kilogram [kg]

- Originally: the mass if 1 dm^3 of water at certain thermodynamic conditions (pressure, temperature, content of D_2O)
- Long time an etalon - international kilogram is used. This seems not particularly consistent with the fact that weighting belongs in principle among the most precise measurements.

Time the second [s]

- Originally: *1/86400* part of the solar day the 1. 1. 1900.
- Recently: on the basis of frequency measurements of the particular spectral line

^{133}Cs: $f = 9\,192\,631\,770\ Hz$

$\lambda = 32{,}6\ mm$

Electric current - the ampere [A]

- With the help of force between two parallel very (~∞) long wires through which the current flows.
- If these wires are 1 *m* apart and the current of 1 A flows in each of them in the same direction there is an attractive force between them 2.10^{-7} *N* per one meter of the wire length exists.
- The current of 1 *A* means that the charge of 1 *C* passes in one second. This represents the charge of 6 242 197 253 433 210 000 elementary (electron) charges.

Absolute Temperature - Kelvin [K]

- The step is the same as in the case of the Celsius degree: The interval between the freezing and boiling points of water is divided into 100 steps. The relation between the Celsius and Kelvin (=absolute or thermodynamic) scale is linear

$$T[K] = 273.15 + T[°C]$$

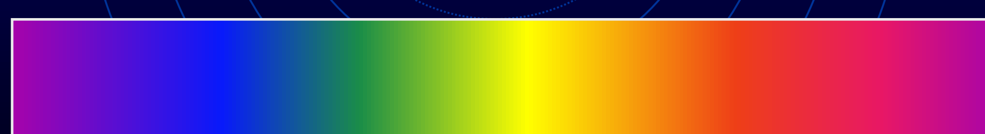
- The Kelvin is defined with the use of the triple point of water which is *273.16 K*
- The temperature is closely related to the kinetic energy of atoms and molecules in matter.

Amount of substance - mole [mol]

- Number of atoms in $0.012\ kg$ of carbon ^{12}C.
- Equal to $N_A = 6.02214379\ 10^{23}$ particles, named after Amedeo Avogadro 1776 - 1856)
- Number that allows for convenient transfer of units from micro-world to our scales world.

Luminous Intensity - candela [cd]

- Candela is a luminous intensity of a monochromatic source with the frequency of 540×10^{12} Hz (yellow-green), the emittance of which in given direction is 1/683 W into 1 steradian.
- Emittance is the energy emitted over one second into the unit of a space angle.



SI multipliers I

kilo	10^3	k
mega	10^6	M
giga	10^9	G
tera	10^{12}	T
peta	10^{15}	P
exa	10^{18}	E

SI multipliers II

milli	10^{-3}	m
micro	10^{-6}	μ
nano	10^{-9}	n
pico	10^{-12}	p
femto	10^{-15}	f
atto	10^{-18}	a

Orders of Length in Nature

• radius of neutron	10^{-15}	m
• radius of atom	10^{-10}	m
• typical length of a virus	10^{-7}	m
• thickness of a sheet of paper	10^{-4}	m
• human's finger	10^{-2}	m
• football playground	10^{2}	m
• Mount Everest	10^{4}	m
• radius of the Earth	10^{7}	m
• the distance from the Earth to the Sun	10^{11}	m
• - " - to Alpha Centauri	10^{16}	m
• - " - to the nearest galaxy	10^{22}	m
• - "- to the most distant galaxy	10^{26}	m

Orders of Time in Nature

• lifetime of some particles	10^{-23}	s
• half-time of nuclei	$10^{-22} - 10^{28}$	s
• flight-time of the light through atom	10^{-19}	s
• - " - a piece of paper	10^{-13}	s
• heart beat	1	s
• a day	10^{4}	s
• a year	10^{7}	s
• human's life	10^{9}	s
• the known history of the humankind	10^{12}	s
• life on the Earth	10^{16}	s
• the expected age of the Universe	10^{22}	s

Orders of Mass in Nature

electron		10^{-30}	kg
proton, neutron		10^{-27}	kg
molecule of DNA		10^{-17}	kg
bacteria		10^{-15}	kg
a flee		10^{-5}	kg
a man		10^{2}	kg
a ship		10^{8}	kg
the Earth	6	10^{24}	kg
the Sun	3	10^{30}	kg
our galaxy		10^{41}	kg

Gonimetric functions

- $cos(\alpha)$ … the first coordinate of the intersection of the oriented angle α with the unit circle
- $sin(\alpha)$ … the second coordinate of the intersection of the oriented angle α with the unit circle
- $tg(\alpha) = sin(\alpha) / cos(\alpha)$
- $cotg(\alpha) = cos(\alpha) / sin(\alpha)$
- $sin^2(\alpha) + cos^2(\alpha) = 1$

Sum Formulas I

- $sin(\alpha+\beta) = sin(\alpha)cos(\beta) + sin(\beta)cos(\alpha)$
- $sin(\alpha-\beta) = sin(\alpha)cos(\beta) - sin(\beta)cos(\alpha)$
- $cos(\alpha+\beta) = cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta)$
- $cos(\alpha-\beta) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$
- $sin(2\alpha) = 2\ sin(\alpha)cos(\alpha)$
- $cos(2\alpha) = cos^2(\alpha) - sin^2(\alpha)$
- $sin^2(\alpha/2) = [1 - cos(\alpha)]/2$
- $cos^2(\alpha/2) = [1 + cos(\alpha)]/2$

Sum Formulas II

- $sin(\alpha)+sin(\beta) = 2sin((\alpha+\beta)/2)cos((\alpha-\beta)/2)$
- $sin(\alpha)-sin(\beta) = 2cos((\alpha+\beta)/2)sin((\alpha-\beta)/2)$
- $cos(\alpha)+cos(\beta) = 2cos((\alpha+\beta)/2)cos((\alpha-\beta)/2)$
- $cos(\alpha)-cos(\beta) = -2sin((\alpha+\beta)/2)sin((\alpha-\beta)/2)$
- *Euler's formula :*

$$exp(-i\alpha) = cos(\alpha) - i\ sin(\alpha)$$

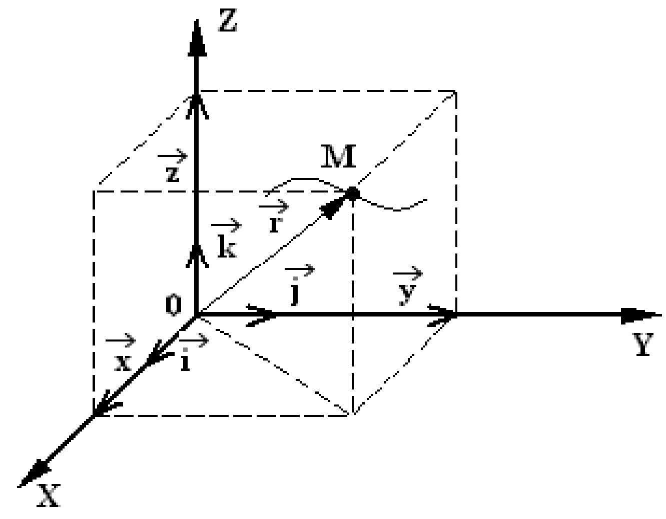
imaginary unit i … $i^2 = -1$

Vectors and Scalars

- A scalar quantity has only its magnitude and unit so it can be expressed as a number e.g. temperature, time, speed, energy…
- A vector quantity has its magnitude, unit and direction and it is used when the direction matters e.g. radius vector, velocity, force, linear momentum, angular momentum, torque…

Rectangular coordinate syst.

$$\vec{r} = \vec{x} + \vec{y} + \vec{z}$$



Each vector can be expressed in a basis of three non-coplanar vectors

Vector operations I

$$r = |\vec{r}| = \sqrt{(x^2 + y^2 + z^2)}$$ …magnitude of a vector

$$\vec{r}_0 = \frac{\vec{r}}{r}$$ … unit vector and

$$\vec{r}_0 = (\cos\alpha, \cos\beta, \cos\gamma)$$ …direction cosines

$$\vec{r} = \vec{x} + \vec{y} + \vec{z} = x \cdot \vec{i} + y \cdot \vec{j} + z \cdot \vec{k}$$ … radius vector in rectangular coordinates

$$\vec{r} = (x, y, z)$$ *i, j, k,* are unit vectors of the basis

Vector operations II

- null vector ... null length, arbitrary direction
- multiplication by a scalar … $k\vec{a} = (ka_1, ka_2, ka_3)$
- an opposite vector when $k = -1$ orientation changes
- addition of vectors … $\vec{c} = \vec{a} + \vec{b}$ $c_i = a_i + b_i$
- subtraction of vectors … $\vec{d} = \vec{a} - \vec{b}$ $d_i = a_i - b_i$

Take care about the units !!!

Vector operations III

- The dot product - the result is a scalar

$$c = \vec{a} \bullet \vec{b} = \left(a_x \cdot b_x + a_y \cdot b_y + a_z \cdot b_z\right)$$

$$c = |\vec{a}|\,|\vec{b}| \cos\gamma$$

⇨ Projection of one vector into the direction of the other

- The cross product - the result is a vector

$$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$c_x = a_y b_z - a_z b_y$$

$$c_y = a_z b_x - a_x b_z$$

$$c_z = a_x b_y - a_y b_x$$

$$|\vec{c}| = |\vec{a}|\,|\vec{b}| \sin\gamma$$

⇨ Projection of one vector perpendicularly to the other one

Vector operations IV

- The trick in using vectors is that rather complicated problem is projected into the axes of a conveniently chosen coordinate system to obtain components, then many but simple manipulations are done with these components and the result vector is constructed back from them.
- The dot product is commutative and convenient to test perpendicularity of vectors.
- The cross product is anti-commutative and convenient to test co-linearity of vectors:
    - If two non-zero vectors are not co-linear they define a plane and the cross product is perpendicular to this plane so that vectors $\vec{a}, \vec{b}, \vec{c} = \vec{a} \times \vec{b}$ make a right-handed system.
    - If the vectors $\vec{a}$ and $\vec{b}$ are co-linear their cross product is always a zero vector.

Calculus in Physics

- Derivative of a function of real variable measures its sensitivity to the change of the function value to the change of variable.

$$f'(x) \equiv \left[\frac{df}{dx}\right]_x$$

- Geometrically it is a slope of a tangent line to the function *f* in the point with the coordinate *x*.
- Derivatives of higher order exist. Example :

  $f(x) = 3x^2+3;\ df/dx = 6x;\ d^2f/dx^2=6;\ d^3f/dx^3=0$

- Derivatives are defined also for functions of more variables.

Physics

2

Kinematics

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Main Topics

- Introduction into mechanics. Kinematics of a particle = mass point.
- Straight-line motion
  - Constant speed
  - Constant acceleration
- Curved line – circular motion
  - Constant speed
  - Constant acceleration
- Motion in 2D and 3D space
- Relative motion

Introduction into Mechanics

- We shall deal with classical mechanics where the objects
  - are much larger than are the typical atomic bond distances in matter
  - move with a speeds considerably lower than *c* light in free space
- Kinematics deals with the description of position and motion of objects but doesn't care about the reasons for their changes.
- Dynamics deals particularly with these reasons. To reach this task, it defines special quantities and takes a special care about those of them that conserve.
- The particle = mass point is a simplified object which is geometrically infinitely small but has non-zero mass. Further we use both words as synonyms.

Kinematics I

- The main reason to study and understand kinematics is that using easily imaginable and illustrative ideas it enables to learn to solve problems which is useful in other areas as well. For example:
  - The first step is to find the real dimension of the problem and employ appropriate coordinate system and quantities.
  - The mathematical tools used in kinematics such as calculus can be used another fields e.g. the physical meaning of integration constants is illustrative and obvious.

Kinematics II

- The position of a particle is described by means of the radius vector which begins in the origin of the coordinate system and ends in the particle:

$$\vec{r} = (x, y, z) = (x_1, x_2, x_3) \qquad [m]$$

- If a body travels a distance *s* over time period *t* its motion can be attributed the average speed $\langle v \rangle = \dfrac{s}{t}$
- If the radius vector changes by $\Delta\vec{r}$ over a period Δt its motion can be described by the vector of average velocity that has obviously a direction. $\langle \vec{v} \rangle = \dfrac{\Delta\vec{r}}{\Delta t}$
- The limit of this quantity for $\Delta t \to 0$ gives the instantaneous velocity

$$\vec{v} \equiv \langle \vec{v} \rangle_{\Delta t \to 0} = \frac{d\vec{r}}{dt} \qquad [m/s]$$

Kinematics III

- The instantaneous velocity is indeed the first derivative of the radius vector and it is tangent to the trajectory of the particle.
- Since also the velocity can change it time we define acceleration: $$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \qquad [m/s^2]$$
- Similarly higher-order accelerations are defined.
- However, the first order acceleration is especially important since it is the only non-zero acceleration if a constant force acts.

Kinematics IV

- Unlike the velocity which is tangent to the trajectory the acceleration doesn't have generally special geometrical relation to it. But it is useful to project the acceleration into two components in the direction of the velocity and perpendicular to it:

$$\vec{a} \equiv \frac{d(|\vec{v}|\vec{\tau}^{o})}{dt} = \frac{dv}{dt}\vec{\tau}^{o} + v\frac{d\vec{\tau}^{o}}{dt} =$$

$$\frac{dv}{dt}\vec{\tau}^{o} + \frac{v^2}{\rho}\vec{n}^{o} = \vec{a}_t + \vec{a}_n$$

- This allows to divide motion into translation and curvilinear depending whether $\vec{a}_n$ is zero or not.

Kinematics V

- In the previous ρ in the curvature radius. The smaller it is the sharper is the curve. For motion on a straight line ρ=∞ and $\vec{a}_n = \vec{0}$.
- Uniform rectilinear motion ρ=∞ and $\vec{a} = \vec{0}$. Axis e.g. x can be identified with the line of motion and we arrive to scalar (1D) problem:

$$v = \frac{dx}{dt} \Rightarrow dx = v\,dt \Rightarrow x(t) = x_0 + vt$$

To know the velocity means only to know the slope of the time dependence of coordinate. To calculate the position at arbitrary time we need to know it at any particular time t_0, often in the beginning. We call this the boundary conditions.

$$x_0 \equiv x(t_0)$$

Kinematics VI

- Uniformly accelerated rectilinear motion: $\vec{a} = \vec{a}_{t0}$

$$a = \frac{dv}{dt} \Rightarrow v(t) = v_0 + at \Rightarrow$$

$$x(t) = x_0 + v_0 t + \frac{a}{2} t^2$$

- Since we integrate twice, two integration constants x_0 and v_0 are necessary to define the motion. If they have the same sing the motion is accelerated if their sing is opposite the motion is decelerated.

Kinematics VII

- In the case of curvilinear motion the normal component of acceleration must be nonzero in curves. Since every part of any curve can be considered as a part of a circle of a certain radius ρ, conclusions from a simple circular motion are valid for any curvilinear motion.
- Suppose that a particle moves on a circle with a constant radius r. Its position can be described by one scalar quantity, either the distance from a certain point along the circumference or an angle of the circulant from particular direction. So, although a circle is a 2D object the circular motion can be described 1D problem.

Kinematics VIII

- In the case of uniform circular motion $\vec{a}_t = \vec{0} \wedge \left|\vec{a}_n\right| = const.$ and the vector of normal component of acceleration is centripetal i.e. points always to the centre of the circle.
- If a particle moves with constant speed *v* along the circumference its movement is periodical since after the period $T = 2\pi r/v$ *[s]* it passes through the same point.
- Circular motion can alternatively be described by number of rotations in a unit of time the frequency $f = 1/T$ $[s^{-1} = Hz]$.
- The speed is

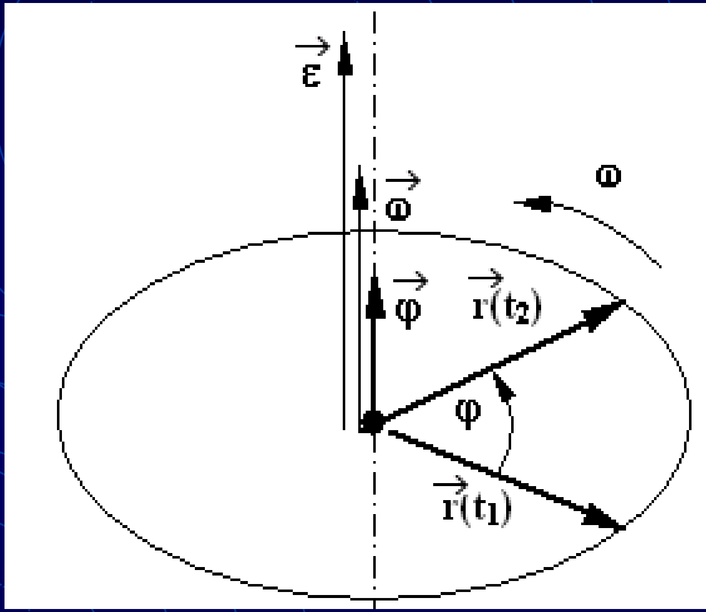
$$v = \frac{ds}{dt} = \frac{d\varphi r}{dt} = \frac{2\pi r}{T} = \omega r$$

Kinematics IX

- In the case of uniformly accelerated circular motion

$$\left|\vec{a}\right|_t = const.$$

- The description still can be simplified to scalar form using either circumference or angle quantities.

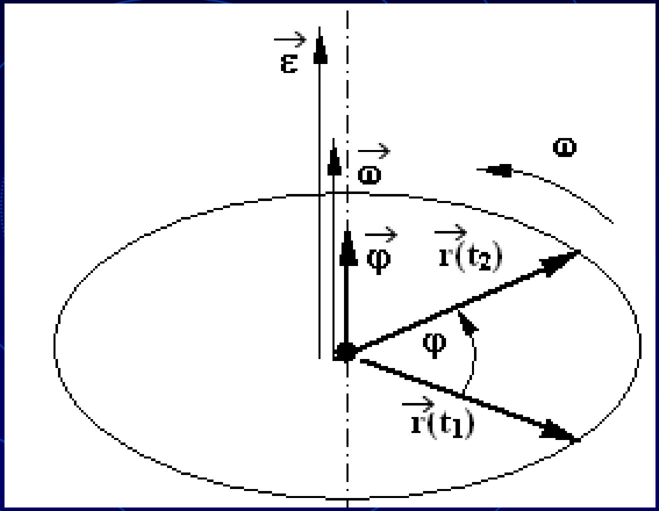


$$\frac{dv_t}{dt} = a_t = \varepsilon . r = \frac{d\omega}{dt} r = \frac{d^2\varphi}{dt^2} r = const.$$

- After integration :
  - $\omega(t) = \omega_0 + \varepsilon\, t$
  - $\varphi(t) = \varphi_0 + \omega_0 t + \varepsilon t^2/2$
- Consider : $x(t) = x_0 + v_0 t + \frac{a}{2} t^2 = r.(\varphi_0 + \omega_0 t + \frac{\varepsilon}{2} t^2)$

Kinematics X

- The simplification which enabled us to use scalar description didn't take into account orientation of the plane of the motion. If we study bodies that move in various planes e.g. motion of several planets we have to use vector description :



- The basis is the definition of the oriented angle $\vec{\varphi}$. It is a normal vector which starts in the origin and from whose endpoint we see the rotation in counter -clockwise = positive sense. The orientation of its time derivatives $\vec{\omega}$, $\vec{\varepsilon}$ and other quantities then develop naturally.

$$\vec{v}_t = \vec{\omega} \times \vec{r}$$

$$\vec{a}_t = \vec{\varepsilon} \times \vec{r}$$

The scalar ≡ dot product

$$Let\ c = \vec{a} \cdot \vec{b}$$

Definition I. (components)

$$c = \sum_{i=1}^{3} a_i b_i$$

Definition II. (projection)

$$c = |\vec{a}|\left|\vec{b}\right| \cos\varphi$$

Can you proof their equivalence?

The vector or cross product I

Let ***c*=*a*.*b***

Definition (components)

$$c_i = \varepsilon_{ijk} a_j b_k$$

The magnitude |***c***|

$$|\vec{c}| = |\vec{a}|\left|\vec{b}\right| \sin\varphi$$

Is the surface of a parallelepiped made by ***a***,***b***.

The vector or cross product II

The vector c is perpendicular to the plane made by the vectors <u>***a***</u> and <u>***b***</u> and they have to form a right-turning system.

$$\vec{c} = \begin{Vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{Vmatrix}$$

ε_{ijk} = {1 (even permutation), -1 (odd), 0 (eq.)}

Physics

3

Dynamics I

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Main Items

- Introduction into Dynamics
- Dynamics of translation motion
- Dynamics of rotation motion
- Conservation of linear momentum, angular momentum and energy

Introduction into Dynamics

- Mechanics would not be complete if it did not study also the reasons why bodies start to move, accelerate, slow down or what causes their trajectories to deviate from a straight line.
- It shows up that if bodies with (constant) mass accelerate or move on a curved trajectories some force (gradient of energy) has to act on them.
- It took very long time to arrive to this simple conclusion since often the forces are not easily visible and/or they can act over a distance.

Dynamics of Translation Motion

- The main new quantities that appear in dynamics of translation motion are the mass, the force, the linear momentum, the kinetic energy and the power. Since these quantities are fundamental in Nature, we shall learn deeply step by step about all of them.

The Mass

- By intuition we understand mass as the measure of the quantity of matter.
- The unit of mass in the SI system is one kilogram.
- It shows up that the inertia mass is equal with the highest precision that can be reached to the gravitational mass. This fact eventually led to the formulation of the general relativity theory.
- Dynamics shows that mass is a measure of the inertia of bodies.
- The heavier the body is, the more effort is needed to change its dynamic status e.g. to accelerate it, to slow it down or to change the direction of its movement.

The Force

- Force is the mediator of interactions between particles. Force is the reason, why bodies start to move, why they accelerate e.g. when falling, why they slow down or change the direction of their movement but also why they are in equilibrium.
- The main unit in the SI system is one newton

$$1N = 1\ kg.\ m\ /\ s^2$$

- Force is a typical vector quantity. The total force acting on a body is a vector sum of all the acting forces. If the total acting force is zero it usually means that there are several non-zero forces acting but they are in equilibrium - they compensate.
- Forces between bodies can be long-range or act directly. But that is, in fact, also long-range since particles of which they consist repel if they are very close. Should their nuclei touch, extreme energies would be needed. The question is what 'to touch' really means.

The Linear Momentum

- The dynamic status of a particle moving on a straight line is given by the vector of its linear momentum (or simply only momentum)

$$\vec{p} = m\vec{v}$$

- This quantity has a direction of the velocity of the particle, is directly proportional to its speed as well as to its mass and it is conserved: The momentum of a particle can change only through an interaction of with other particles by the means of forces. But as we study in detail later the total momentum of a system of particles is conserved.

The Newton's Laws

- Isaac Newton 1642-1727 ingeniously summarized the knowledge of classical mechanics into three laws:

  1. The Law of Inertia

  2. The Law of Force

  3. The Law of Action and Reaction

  These laws have to be modified only beyond the classical physics e.g. in micro-world.

The Law of Inertia

- If the total force acting on a body is zero the body is in the rest of moves with a constant velocity.
  - More precisely: If the total force acting on a body is zero its linear momentum stays constant.
  - Special movements such as those with changing mass are also taken into account in this formulation.

The Law of Force

- The total force acting on a body is equal to the change of its linear momentum in time

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad (= \frac{d(m\vec{v})}{dt})$$

- It is often the case (but not generally !) that the mass of the body stays constant than simpler equation holds:

$$\vec{F} = m\vec{a}$$

  - Note that we are dealing with vector equations which are valid for each of their components e.g.

$$F_y = \frac{dp_y}{dt} = \frac{d(mv_y)}{dt})$$

The Law of Action and Reaction

- If the body 1 acts on the body 2 by the force $\vec{F}_{12}$,
- and the body 2 acts on the body 1 by the force $\vec{F}_{21}$, then

$$\vec{F}_{12} = -\vec{F}_{21}$$

- Both forces have equal magnitude but opposite orientation.
  - BUT each of these forces belongs to a different body so they can't be generally added and thereby cancelled.
  - They cancel only if some additional mechanical connection exists between the bodies.

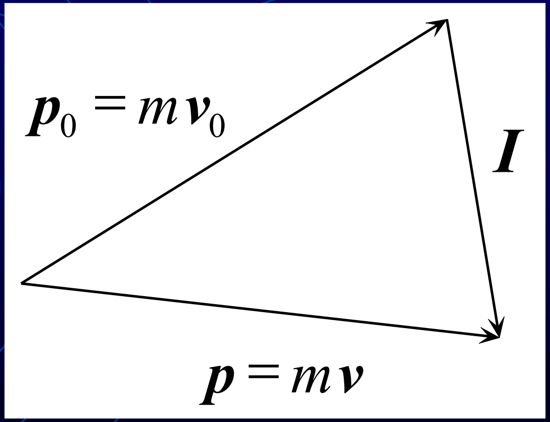
Time Action of the Force – The Impulse

- Let us define the impulse of the force constant over the (very short) time interval dt as: $\vec{I} \equiv \vec{F}dt$
- Then a simple integration of the 2nd newton's law gives:

$$\vec{I} = \int_0^t \vec{F}dt = \int_0^t m\frac{d\vec{v}}{dt}dt = \int_0^t m d\vec{v}$$

or for finite time interval Δt:

$$\vec{I} = \vec{F}\,\Delta t = \vec{p} - \vec{p}_0$$



Space Action of the Force – The Mechanical Work

- Let us define the work done by the force constant over some (very short) distance $d\vec{r}$: $dW \equiv \vec{F} \cdot d\vec{r}$
- Apparently, work is the effect of a force into the direction of the movement. Let this be dx :

$$dW = F_x dx = m\frac{dv}{dt}dx = m\frac{dx}{dt}dv = mv\,dv = d\,\frac{mv^2}{2} = dE_k$$

- The element of work is then equal to the change of a new quantity – the kinetic energy that similarly as the linear momentum describes a sort of dynamic status of a body and also is conserved.

Linear Momentum versus Kinetic Energy

- We have encountered two different descriptions of dynamical status of a body the linear momentum and the kinetic energy. Both depend on its mass and velocity, however the kinetic energy depends on the square of velocity (which is equal to the square of speed) and so it is a scalar quantity.
  - We can easily imagine a system with zero linear momentum but nonzero kinetic energy, e.g. two spheres rolling toward each other with the same speed or a macroscopic amount of gas (particles).
  - Also, linear momentum is connected to motion that is conserved while part of kinetic energy can change into the same amount of energy of different form but the mechanical part can disappear only if it appears somewhere else in the system or the original linear momentum was zero.

The Power

- The power is the speed with which the mechanical work is done:

$$P = \frac{dW}{dt}$$

- As with speed we can use the average power over some time period or the immediate power.
- We can also derive an important formula when a constant force acts:

$$P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

System of Particles

- So far we have dealt with the mechanics of a mass point or a particle. This simplification is convenient to introduce basic quantities of dynamics but can be also useful to solve certain real-life problems.
- More general system can be considered as a system of particles which may interact in some way.
- What has to be counted in is of course a crucial thing. For example: A free falling body hits the ground then stays at rest. What happened to its momentum which 'disappeared' yet it should have been conserved?

The center of Mass I

- The whole system of particles can be represented by the center of mass $\vec{r}_t$, where the whole mass of the system could be concentrated $m = \sum_i m_i$

$$\vec{r}_t = \frac{1}{m} \sum_i m_i \vec{r}_i$$

- This definition holds in components. This can be used if the dimension of a problem is less than 3 :

$$x_t = \frac{1}{m} \sum_i m_i x_i \quad , \quad y_t = \frac{1}{m} \sum_i m_i y_i \quad , \quad z_t = \frac{1}{m} \sum_i m_i z_i$$

*The center of Mass II

- It comes from integration of the equation for the total linear momentum : $m\vec{v} = \sum_i m_i \vec{v}_i$

$$\vec{r} = \vec{r}_t + \vec{c} = \frac{1}{m}\sum_i m_i \vec{r}_i + \vec{c} \Rightarrow \qquad \vec{r}_t = \frac{1}{m}\sum_i m_i \vec{r}_i$$

The center of Mass III

- The center of mass :
  - Doesn't depend on the choice of the coordinate system. But a good choice can simplify considerably its calculation.
  - It is in the intersection of the elements of symmetry. This helps us to choose the coordinate system.
  - For the bodies with rotation symmetry the Papp's theorem can be used :

    The path of the center of mass times surface = volume.
  - If the body consists from parts the center of mass can be found from their centers of mass $\vec{r}_i$ and masses m_i.
  - This is in fact rearrangement of the terms in the definition formula.

The center of Mass IV

- Let's move the origin to the center of mass :

$$\vec{s}_i = \vec{r}_i - \vec{r}_t$$

- Then : $$\sum_i m_i \vec{s}_i = \sum_i m_i \vec{r}_i - \sum_i m_i \vec{r}_t = m\vec{r}_t - m\vec{r}_t = \vec{0}$$
- This equation can be used to prove important properties of the center of mass : Rotation of a body around an arbitrary axis passing through the center of mass and translation of this center are mutually independent quantities.

The First Impulse Law I

- The sum of forces acting on an *i-th* point particle can be decomposed into the sum of internal forces which stem from the interactions of the particles which are part of the system and the sum of external forces. According to the second Newton's law :

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i = \vec{F}_i^{\,I} + \vec{F}_i^{\,E}$$

The First Impulse Law II

- The total linear momentum of the system is the sum of all linear moments :

$$\vec{P} = \sum_i \vec{p}_i$$

- Then :

$$\frac{d\vec{P}}{dt} = \sum_i (\vec{F}_i^I + \vec{F}_i^E) \overset{!}{=} \sum_i \vec{F}_i^E = \vec{F}^E$$

The First Impulse Law III

- The time change of the total linear momentum is the sum of the external forces.
- In other words the total linear momentum can be influenced only by external forces.
- This is an important consequence of the law of action and reaction. Due to this the sum of all internal forces is equal to zero :

$$\sum_i \vec{F}_i^{\,I} = \sum_{i,j} (\vec{F}_{i,j}^{\,I} + \vec{F}_{j,i}^{\,I}) = \sum_{i,j} (\vec{F}_{i,j}^{\,I} - \vec{F}_{j,i}^{\,I}) = \vec{0}$$

The scalar ≡ dot product

$$Let\ c = \vec{a} \cdot \vec{b}$$

Definition I. (components)

$$c = \sum_{i=1}^{3} a_i b_i$$

Definition II. (projection)

$$c = |\vec{a}|\left|\vec{b}\right| \cos\varphi$$

Can you proof their equivalence?

The Center of Mass I

Where is the center of mass of four little spheres of mass 1, 2, 3 and 4 kg laying on a straight line always *1 m* apart?

The straight line is apparently an axis of symmetry so we coincide it with one axis e.g. x. Then we chose the origin conveniently in the center of one of the spheres e.g. the first :

$$x_T = \frac{\sum m_i x_i}{\sum m_i} = \frac{1.0 + 2.1 + 3.2 + 4.3}{10} = 2$$

The center of mass lies in the center of the third sphere. We can test that it will stay there even if we change the origin. A good choice of a the coordinate system doesn't change the position of the center of mass makes calculations easier!

The Center of Mass II

Where is the center of mass of four little spheres of mass 1, 2, 3 and 4 kg laying in the corners of a square with the side *1m* ?

Now the system has a plane of symmetry so we know the center of mass lies in this plane as well and calculate its two coordinates. We coincide the with two sides of the square so one of the spheres lies in the origin. So the coordinates of the spheres are e.g.: 1:[0,0], 2:[1,0], 3:[0,1] a 4:[1,1]. Then the center of mass is $[0.6, 0.7]$:

$$x_T = \frac{\sum m_i x_i}{\sum m_i} = \frac{2\cdot 1 + 4\cdot 1}{10} = 0.6$$

$$y_T = \frac{\sum m_i y_i}{\sum m_i} = \frac{3\cdot 1 + 4\cdot 1}{10} = 0.7$$

The Center of Mass III

Where is the center of mass of four little spheres of mass 1, 2, 3 and 4 kg laying in some corners of a cube with the side *1m* ?

Mějme nyní koule o hmotnosti 1, 2, 3 a 4 kg v některých rozích krychle o straně 1 m. Kde je těžiště tohoto systému?

Now the problem is 3D but it is still convenient to define a special coordinate system so the coordinates of the spheres are e.g. : 1:[0,0,0], 2:[1,0,0], 3:[0,1,0] a 4:[0,0,1]. Then the center of mass is [0.2, 0.3, 0.4] :

$$x_T = \frac{2\cdot 1}{10} = 0.2;\ y_T = \frac{3\cdot 1}{10} = 0.3;\ z_T = \frac{4\cdot 1}{10} = 0.4$$

Due to the choice of coordinate system the calculation was very simple.

The C. of M. IV Papp's Theorem

Where is the center of mass of half of the circular disk the radius of which was *a*?

The body has a plane of symmetry and a two-fold axis which we coincide with the axis *x* and on this axis the center of mass should lie. The axis *y* will be the straight line of the cut of the original disk. Its center is also the origin.

If we rotate our body one turn along the axis y we get a sphere.

If desired coordinate is x_T then according to the Papp's theorem:

$$\frac{\pi a^2}{2} 2\pi x_T = \frac{4\pi a^3}{3} \Rightarrow x_T = \frac{4a}{3\pi}$$

The Center of Mass V

Calculating the same using integrals is considerably harder, even if we use the best possible system of polar coordinates :

$$x_T = \frac{2\int_0^a \int_0^{\pi/2} r^2 \cos\varphi \, d\varphi \, dr}{2\int_0^a \int_0^{\pi/2} r \, d\varphi \, dr} = \frac{\int_0^a r^2 \, dr \int_0^{\pi/2} \cos\varphi \, d\varphi}{\int_0^a r \, dr \int_0^{\pi/2} d\varphi} =$$

$$\frac{[\frac{r^3}{3}]_0^a [\sin\varphi]_0^{\pi/2}}{[\frac{r^2}{2}]_0^a [\varphi]_0^{\pi/2}} = \frac{4a}{3\pi}$$

Of course, integration allows to calculate problems with much lower symmetry than the Papp's theorem needs.

Physics

4

Dynamics II

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Main Topics

- Closer to the reality :
  - System of particles
  - Rigid body
  - Angular momentum, torque
  - Dynamics of rotation motion
  - The second impulse theorem
  - Centre of gravity, moment of inertia, Steiner's law
  - Translation and rotation effect of a force

Angular momentum -conservation laws

- From dynamics of a particle it follows that if the resulting force is zero the linear momentum and kinetic energy of the particle is conserve.
- Translation motion can be considered as rotation around the origin and it is possible to define its – angular momentum :

$$\vec{b} \equiv \vec{L} = \vec{r} \times \vec{p}$$

- This quantity is also conserved. It is conserved also in the case of non-zero force if it is central as it is e.g. in the case of planetary movement.

Dynamics of Rotation Motion I

- A force can change both the translation and the rotation motion of bodies. In the latter case it is important how the force acts.
  - Let's have a horizontal rod that can rotate around horizontal axis perpendicular to one end of the rod. In the distance r from the axis there is a mass point m.
  - In the distance ρ we act by a force F to compensate the weight of the mass point so the rod is in equilibrium.

Dynamics of Rotation Motion II

- Since the weight acts down the vertical component of our force has to be equal to it :

  $F_k = F sin(\alpha)$.
- It can be proved experimentally that :
  - The weight of the mass point $G = mg$ bodu is supported in the axis and by our force: $G = F_0 + F_k$.
  - Distribution of the forces is indirectly proportional to the distance of the supporting forces :

    $F_0 r = F_k (\rho - r)$.
- So : $G r = F_k \rho$

Dynamics of Rotation Motion III

- It can be seen that the rotation effect of a force depends on the distance from the axis where it acts and also on its direction related to the direction from the axis - to the point where the force acts.
- Totally the rotation effect of a force is given by its torque :

$$\vec{T} \equiv \vec{M} = \vec{r} \times \vec{F}$$

The origin is in the intersection of the axis and the plane of rotation.

Dynamics of Rotation Motion IV

- Let a constant torque act on a particle. Then using the second Newton's law we can write:

$$\vec{T} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = d(\vec{r} \times \vec{p})/dt = d\vec{b}/dt$$

- The torque is equal the time derivative of the angular momentum.
- This is the most general formulation of the second Newton's law for the rotation motion.

Dynamics of Rotation Motion V

- In the case the body has constant mass and geometry (distribution of the mass) it is convenient to define the moment of inertia (related to a particular axis):

$$J = \Sigma m_i r^2_i$$

and to write :

$$\vec{T} = J\vec{\varepsilon}$$

- The meaning of this can be illustrated similarly as before :

Dynamics of Rotation Motion VI

- A mass body m, lays on a rod at the distance r from the rotation axis which is this time vertical :
  - The force F lays in horizontal plane and acts in the distance ρ from this axis under an angle α with the rod.

    Using the previous :

$$F_k\,\rho = F\,sin(\alpha)\,\rho = r\,m\,a = r^2\,m\,\varepsilon\,.$$

  - If we put more mass points on the rod then after taking out the angular acceleration, which is the same for all the points, we find out the particularly the product $m_i\,r^2{}_i$ is additive :

$$F\sin(\alpha)\rho = r_1 m_1 a_1 + r_2 m_2 a_2 + \ldots = \varepsilon \sum m_i r_i^2$$

The Second Impulse Law I

- If we have a system of particles we can consider rotation effect of the force acting on the *i-th* particle with respect to arbitrary fixed point *O* :

$$\frac{d\vec{b}_i}{dt} = \vec{T}_i = \vec{r}_i \times (\vec{F}_i^{\,I} + \vec{F}_i^{\,E}) = \vec{T}_i^{\,I} + \vec{T}_i^{\,E}$$

The Second Impulse Law II

- The total rotation momentum is the vector sum of rotation momentum of all the particles related to the same fixed point O :

$$\vec{B} = \sum_i \vec{b}_i = \sum_i \vec{r}_i \times m_i \vec{v}_i$$

  - When summing over the whole system we can employ the law of action and reaction.

The Second Impulse Law III

- Then the time change of the rotation momentum is equal to the resulting torque due to external forces related to the same fixed point O for all the points:

$$\frac{d\vec{B}}{dt} = \sum_i \vec{T}_i^I + \sum_i \vec{T}_i^E = \sum_i \vec{T}_i^E = \vec{T}^E$$

Consequences of the Impulse Laws

- If the sum of external forces is zero then the total linear momentum is constant.
- If the sum of torques of external forces is zero then the total angular momentum is constant.
- External forces have both translation and rotation effect according the way they act related to the centre of gravity of the body (or of the system of particles).

Example – Collision of Bodies I

- Central collision – bodies are spherical, they are no external forces. .
- Before the collision the bodies have masses m_i, and velocities v_i.
- After the collision they have velocities u_i.
- According to the first impulse law the total linear momentum is conserved :

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

- Collisions exist between two limits – totally inelastic where the bodies move together after the collision $u_1 = u_2 = u$, part of energy changes to non-mechanical :

$$m_1 v_1 + m_2 v_2 = m_1 u + m_2 u \Rightarrow u = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

- Totally elastic – also the kinetic energy is conserved :

$$m_1 v_1^2 + m_2 v_2^2 = m_1 u_1^2 + m_2 u_2^2$$

Collision of Bodies II

$$m_1(v_1^2 - u_1^2) = m_2(u_2^2 - v_2^2)$$

$$m_1(v_1 - u_1) = m_2(u_2 - v_2)$$

after diving the equations

$$v_1 + u_1 = u_2 + v_2$$

we arrive to the solution

$$u_1 = \frac{2m_2 v_2 + (m_1 - m_2)v_1}{(m_1 + m_2)}$$

Motion with changing mass. Rocket Motion I

- Let's assume a body with the mass m and velocity $\vec{v}$ collides nonelactically with a body dm and $\vec{u}$:

$$d\vec{P} = \vec{P}_P - \vec{P}_A = (m + dm)(\vec{v} + d\vec{v}) - (\vec{u}\, dm + m\vec{v}) \Rightarrow$$

$$\vec{F}^E = \frac{d\vec{P}}{dt} = m\frac{d\vec{v}}{dt} - (\vec{u} - \vec{v})\frac{dm}{dt} \Rightarrow$$

$$m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}^E + (\vec{u} - \vec{v})\frac{dm}{dt} = \vec{F}^E + \vec{v}_{rel}\frac{dm}{dt}$$

Rocket Motion II

- The linear momentum of a body can change as a response to an external force or by receiving or emitting mass with certain non-zero relative velocity.
- We can assume linear movement :

$$m\frac{dv}{dt} = ma = F^E + (u-v)\frac{dm}{dt} \Rightarrow$$

$$dv = \frac{F^E}{m}(t)dt + v_{rel}\frac{dm}{m}$$

Rigid Body I

- In the previous parts when we introduced quantities important for rotations e.g. torque we needed fictive bodies like rigid rods with negligible mass which would transfer force and torque.
- This is an important category of bodies called rigid bodies. They are not deformed by acting of forces.

Rigid Body II

- In reality it means that deformations which are always present for real bodies can be neglected from the standpoint of the current problem.
- For these bodies it is easy to decompose action of an external force to the translation and rotation effect. This further depends on some supplement conditions.
- For these bodies the moment of inertia doesn't change and has unambiguous meaning.

Rigid Body III

- Neither the translation nor rotation effect on rigid body changes if :
  - into any arbitrary point we introduce two forces with the same magnitude but opposite orientation
  - we shift any acting force on the straight line of its action
  - $\Rightarrow$ on any straight line we place two forces with the same magnitude but opposite orientation

Rigid Body IV

- The effect of force which acts in the straight line passing through the centre of mass is purely translational
- The effect of two forces with the same magnitude but opposite orientation acting in two arbitrary parallel straight lines is purely rotational
- One force can’t have purely rotational effect

Rigid Body V
Steiner's Law I

- For rigid bodies an important quantity is the moment of inertia is :

$$J = \Sigma m_i r^2{}_i$$

- From the properties of the centre of mass the Steiner's law follows:

$$J_a = J_t + ma^2$$

- Here J_t is the moment of inertia related to the axis passing through the centre of mass and J_a is the moment of inertia related to an axis parallel to it passing in the distance a.

Rigid Body VI
Steiner's Law II

- The radius vector of *i*-th point can be expressed using its radius vector in the system where the origin is the centre of mass : $\vec{r}_i = \vec{r}_t + \vec{s}_i$
- So : $$\sum_i m_i r_i^2 = \sum_i m_i (\vec{r}_t + \vec{s}_i)(\vec{r}_t + \vec{s}_i) =$$

$$r_t^2 \sum_i m_i + 2\vec{r}_t \sum_i m_i \vec{s}_i + \sum_i m_i s_i^2$$

The term in the middle must be zero.

Rigid Body VII
Steiner's Law III

- It can be readily seen that the moment of inertia related to the axis passing through the center of gravity is the smallest of the all parallel axes.
- If the sum of all torques acting on a rigid body is zero, the body uniformly rotates along the axis passing through the center of mass or remains at rest.

Rigid Body VIII
Statics

- If the sum of all forces acting on a rigid body is zero, the body moves uniformly along the straight line or remains at rest.
- Statics studies the conditions at which bodies remain at rest. Generally all forces and all torques (= all their components) must be compensated

Rigid Body IX
Kinetic energy

- From the above we see that the total kinetic energy of a rigid body generally has a translation and rotation components :

$$E_k = \tfrac{1}{2} m v^2 + \tfrac{1}{2} J \omega^2$$

Rigid Body X
Translation and Rotation

- The corresponding general formulas for dynamics of translation and rotation motion are :

$$\vec{p} = m\vec{v} \qquad \vec{b} \equiv \vec{L} = \vec{r} \times \vec{p}$$

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \vec{T} \equiv \vec{r} \times \vec{F} = \frac{d\vec{b}}{dt}$$

- If neither mass nor its geometry changes it makes sense to use the moment of inertia : $J = \Sigma m_i r^2{}_i$ then relations can be simplified.

Rigid Body XI
mass ~ moment of inertia

- The formulas for the rotation motion have the moment of inertia at the place corresponding to mass in the formulas for the translation motion :

$$\vec{p} = m\vec{v} \qquad \vec{b} = J\vec{\omega}$$

$$\vec{F} = m\vec{a} \qquad \vec{T} = J\vec{\varepsilon}$$

$$E_k = \frac{mv^2}{2} \qquad E_k = \frac{J\omega^2}{2}$$

The scalar ≡ dot product

$$Let \;\; c = \vec{a} \cdot \vec{b}$$

Definition I. (components)

$$c = \sum_{i=1}^{3} a_i b_i$$

Definition II. (projection)

$$c = |\vec{a}| \left|\vec{b}\right| \cos\varphi$$

Can you proof their equivalence?

^

The vector or cross product I

Let ***c***=***a***.***b***

Definition (components)

$$c_i = \varepsilon_{ijk} a_j b_k$$

The magnitude |***c***|

$$|\vec{c}| = |\vec{a}|\left|\vec{b}\right| \sin \varphi$$

Is the surface of a parallelepiped made by ***a***,***b***.

The vector or cross product II

The vector c is perpendicular to the plane made by the vectors ***a*** and ***b*** and they have to form a right-turning system.

$$\vec{c} = \begin{Vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{Vmatrix}$$

ε_{ijk} = {1 (even permutation), -1 (odd), 0 (eq.)}

Two weights on a pulley I

Let's have two bodies hanging from a cylindrical pulley of the mass m_3 and radius r. The body on the left has a mass m_1 and the one on the right m_2 ($<m_1$). How the system moves? Apparently the acceleration of body m_1 is $_\mathrm{a}$ in the downward direction let's find this acceleration.

For the forces t_1 and t_2 which act on the circumference of the pulley we can write :

$$m_1 g = t_1 + m_1 a \Rightarrow t_1 = m_1 g - m_1 a$$

$$t_2 = m_2 g + m_2 a$$

$$(t_1 - t_2) r = J\varepsilon$$

Two weights on a pulley II

If the influence of the pulley could be neglected then :

$J \approx 0 \Rightarrow t1 = t2 \Rightarrow$

$$a = \frac{m_1 - m_2}{m_1 + m_2}$$

if not then :

$$J = \frac{m_3 r^2}{2}; \quad a = r\varepsilon$$

Two weights on a pulley III

After he substitution :

$$t_1 = t_2 + \frac{J\varepsilon}{r} = t_2 + \frac{m_3 r^2 a}{2r^2} = t_2 + \frac{m_3 a}{2}$$

and reorganizing :

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{m_3}{2}}$$

Two weights on a pulley IV

The same result can be obtained from the conservation of the energy: let's assume that the body m_1 moves down during Δt by Δh The loss of its potential energy must be equal the gain of the kinetic of the whole system (both bodies and the pulley) and the gain of the potential energy of the second body :

$$(m_1 - m_2) g \Delta h = (m_1 + m_2) \frac{v^2}{2} + J \frac{\omega^2}{2}$$

$$(m_1 - m_2) g \Delta h = (m_1 + m_2 + \frac{m_3}{2}) \frac{v^2}{2}$$

Finally, we derive by time and rearrange :

$$(m_1 - m_2) g v = (m_1 + m_2 + \frac{m_3}{2}) \frac{2 v a}{2}$$

The moment of inertia of a thin rod

Let's calculate the moment of inertia of a thin homogeneous rod with the cross section *S,* length L and density ρ related to the axis perpendicular to the rod, intersecting it at its end :

$$J_L = S\rho\int_0^L x^2 dx = \tfrac{1}{3} S\rho L^3 = m\frac{L^2}{3}$$

Taking the second axis parallel to the previous axis but intersecting it in the center of mass we can verify Steiner's law:

$$J_T = 2S\rho\int_0^{L/2} x^2 dx = \tfrac{1}{12} S\rho L^3 = m\frac{L^2}{12}$$

The moment of inertia of a cylinder

To calculate the J of homogeneous cylinder with the length L and radius R and density ρ we use conveniently the polar coordinates :

$$J = 4\rho L \int_0^{\pi/2}\int_0^R r^3 dr\, d\varphi = \frac{\rho L \pi R^4}{2} = m\frac{R^2}{2}$$

Note that the central angle doesn't have to be necessarily 2π but arbitrary. So, for instance, for the J of a quarter of a cake along the axis passing through its tip the same formula can be used. Only now the mass is just one quarter of that of the original cake.

Motion with changing Mass I

75 kg/s of sand with zero component of horizontal velocity falls on a conveyor belt running horizontally with the speed of 2.2 m/s.

What force is necessary to keep the belt moving?

What power must the motor that drives the belt have?

We start from the equation for motion with changing mass :

$$F = ma - (u - v)\frac{dm}{dt} = 0 - (0 - v)\frac{dm}{dt} = v\frac{dm}{dt}$$

Motion with changing Mass II

- The force necessary to keep the belt moving with constant speed is then : 2.2*75 = 165 *N*.
- The power necessary :

$$P = Fv = v^2 \frac{dm}{dt} = 363\,W$$

Interestingly, just half of this power goes to the increase of the kinetic energy of the sand.

Motion with changing Mass III

- A fully loaded rocket with the mass of 21000 kg of which 15000 kg is fuel is starting from the Earth directly up. 190 kg of the burned fuel with the speed of 2800 m/s leaves it every second.

What is the thrust of the motors?

What is the final force minus the force of gravity at the moment of start and just before the fuel is burned up?

How long it takes the fuel to burn up?

What us the final speed of the rocket?

Motion with changing Mass IV

The thrust :

$$F_T = v\frac{dm}{dt} = -2800 \cdot -190 = 5.32 \cdot 10^5 \ N$$

The initial and end force :

$$F_0 = v\frac{dm}{dt} - m_0 g = 3.2 \cdot 10^5 \ N$$

$$F_E = v\frac{dm}{dt} - m_E g = 4.7 \cdot 10^5 \ N$$

Motion with changing Mass V

We neglect the air resistance and suppose that during the 79 s flight of the rocket the free fall acceleration g is constant. Then :

$$m\frac{dv}{dt} = mg + v_r \frac{dm}{dt} \Rightarrow dv = g\,dt + v_r \frac{dm}{m} \Rightarrow$$

$$\int_{v_0}^{v} dv = g\int_0^t dt + v_r \int_{m_0}^{m(t)} \frac{dm}{m} \Rightarrow v(t) = v_0 - gt + v_r \ln\frac{m(t)}{m_0}$$

After substituting for the final time and mass the final speed is v = 2730 m/s.

Motion with changing Mass VI

The rocket hasn't reached even the first cosmic speed so if it hadn't got another stage it would eventually fall back to the Earth.

The assumption about the free fall acceleration is OK since at the height of 100 km it is only 1.5% less than on the Earth.

But the resistance of the air can't be neglected taking into account the speeds involved.

Two special cases of an elastic collision I

1) $m_1 = m_2, v_2 = 0 \rightarrow$

$u_1 = 0, u_2 = v_1$

Particles change their velocities. Should the collision have occurred in a black-box and we can't distinguish the particles, we don't have any means to find out whether the particles had collided or not.

2) $2m_1 = m_2, v_2 = 0 \rightarrow$

$u_1 = -v_1/3, u_2 = 2v_1/3$

We can easily verify that :

$u_1 + v_1 = u_2 + v_2$

Physics I

5

Gravitation

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Main Topics

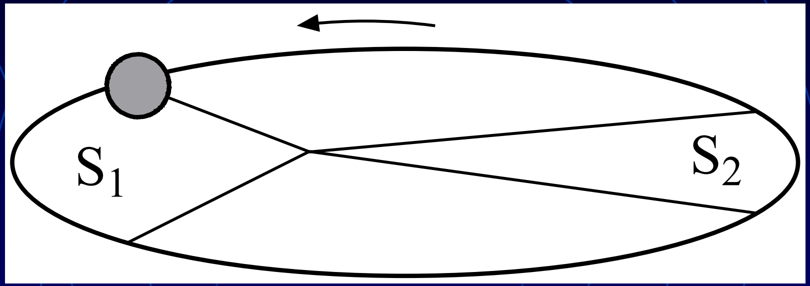
- Kepler's Laws
- Newton's Law of Universal Gravitation
  - G. Field General / Near the Earth's Surface
  - Planetary Motion
- Conservative Fields
  - Potential Energy and Potential
  - The Relation of Intensity and Potential

Introduction into Gravitation

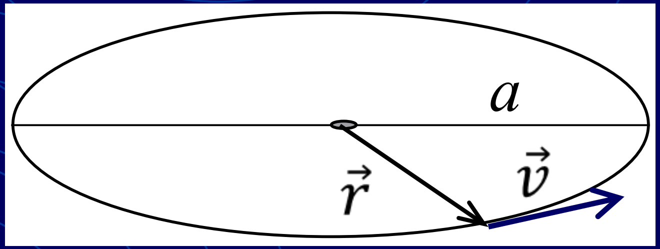
- We encounter the first long-range force - the force of gravitation. It is connected to an important property of nature - the mass. Bodies that have mass effect each other (act on each other by force) without touching each other.
- The celestial mechanics works on the basis of the gravitational force.
- The laws of gravitation are a generalization of astronomical observations, which are typical kinematic and took ages.
- The exact description of gravitation was started by very accurate measurements of Tycho Brahe (1545-1601) were summarized into three laws of planetary motion by Johannes Kepler (1571-1630).

Kepler's Laws

1. Planets move around the Sun on elliptical (almost circular) trajectories. The Sun is in their common focal point.



2. The areal velocity of any planet in any point of its trajectory is constant:



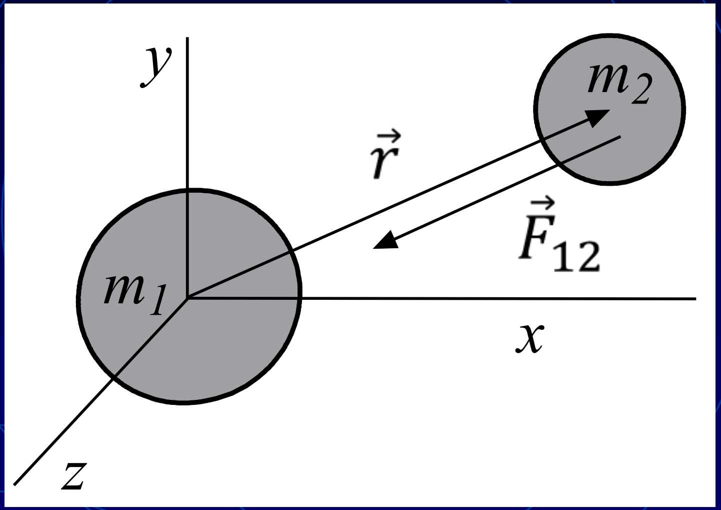
$$\vec{w} = \frac{\vec{r} \times \vec{v}}{2}$$

3. Comparing the motion parameters of two different planets:

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

Newton's Law of Universal Gravitation

- The three Kepler's laws were further generalized into one law of gravitation by Isaac Newton: Any two mass-points act on each other by attractive force acting along their join. The force is proportional to each of the masses and indirectly proportional to the square of their distance:



$$\vec{F}_{12}(\vec{r}) = -\kappa \frac{m_1 m_2}{r^2} \vec{r}^{\,0}$$

- Here F_{12} is the force acting on m_2, whose position is $\vec{r}$ due to the existence of m_1 in the origin.
- The reaction of this force acts on the mass m_1.

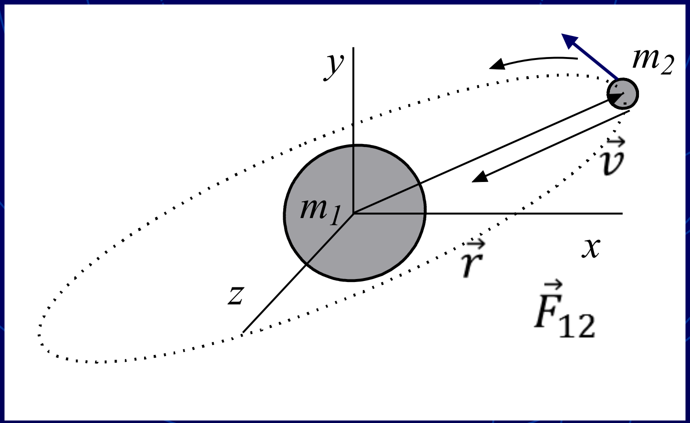
Newton's G. Law – notes I

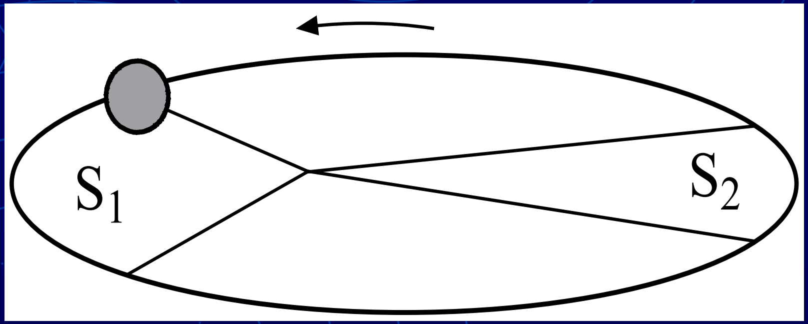
- The '-' sign means that the force is always attractive.
- $\kappa = 6.67\ 10^{-11}\ Nm^2kg^{-2}$ … is the universal gravitational constant
- Any two mass-points attract each other but the forces between bodies of common masses ~kg are almost negligible thereby difficult to sense or measure.
- If more masses act of each other the principle of superposition holds ≡ the force between any two mass points doesn't depend on distribution of any other masses, even if it was between them.

Newton's G. Law – notes II

- The areal velocity is defined as : $\vec{w} = \frac{\vec{r} \times \vec{v}}{2}$
- Apparently it is proportional to the angular momentum :

$$\vec{b} = 2m\vec{w}$$





So the conservation of the areal velocity is equivalent the conservation of the angular momentum –> the force of gravity is central.

Newton's G. Law – notes III

- Gravitational field can be roughly imagined as 'information' that mass-bodies transmit to their surroundings and that may change the properties of space/vacuum.
- this information includes their size and position
- spreads with the speed of light in vacuum
- Other mass-points are sensitive to this information a force acts on them
- Generally – if particle is a source of a certain kind of field it is also sensitive to this particular field.

Gravitational field – intensity $\vec{E}$

- Gravitational field is a vector field so it can be fully characterized by three components of some vector. This could be the force $\vec{F}(m,\vec{r})$ acting on some testing mass m.
- It is more convenient to divide this force by the testing mass and get the gravitational intensity $\vec{E}$ which is an unequivocal property of gravitational field.

$$\vec{E}(\vec{r}) = \frac{\vec{F}(m,\vec{r})}{m} \Leftrightarrow \vec{F}(m,\vec{r}) = \vec{E}(\vec{r}) \cdot m$$

Gravitational field – intensity II

- e.g. in the case of point mass

$$\vec{E}_1(\vec{r}) = \frac{\vec{F}_{12}(\vec{r})}{m_2} = -\kappa \frac{m_1}{r^2} \vec{r}^{\,0}$$

- The intensity is a force that would act on unit mass
- So, it is independent on the test mass

Gravitational intensity close to the Earth surface I

- Gravitational field close to the surface of the Earth is characterized by intensity which we call the gravitational acceleration.

$$\vec{E}_1(\vec{r}) = -\kappa \frac{M}{R^2}\vec{r}^{\,0} = -a_g \vec{r}^{\,0}$$

- After corrections of the gravitational acceleration $a_g = 9.83$ ms^{-2} for various effects, mainly the rotation of the Earth we get the measurable free-fall acceleration. Its average value is $g = 9.81\ ms^{-2}$.

Gravitational intensity close to the Earth surface II

- Minor changes of the free fall acceleration due to mass inhomogeneities of the surface layers of the Earth can be used e.g. for geological exploration.
-
- The product κM present in the gravitational formulas cause that masses can be evaluated only relatively to each other. To evaluate masses in common mass units kg, κ must be measured in laboratory e.g. using Cavendish scales. That is, in fact, also relative measurement but using the proper mass etalon.

Satellite Motion I

- Generally two bodies rotate around their common center of gravity.
- If the satellite is considerably lighter than the central body the center of gravity can be identified with the center of the central body. In this case the satellite rotates around the central body.
- For simplicity let us suppose that the orbit is circular, then the centripetal force is accomplished by gravitation:

$$\frac{mv^2}{r} = \frac{\kappa\, mM}{r^2}$$

Satellite Motion II

- From this relation we can for instance deduce the orbiting speed :

$$v = \sqrt{\frac{\kappa M}{r}}$$

- If both masses are comparable the common center of mass is somewhere on their join and both masses rotate around it. So even the 'central' body moves due to the rotation of its satellite.
- Using this principles tidal forces or search for exo-planets can be explained.

1st Cosmic speed

- 1st Cosmic speed of a body (a planet) is the orbiting speed just above the surface of this body.
  - This motion is only possible if the body doesn't have an atmosphere. Otherwise the satellite would be slowed down and probably burn.
  - In the case of Earth this speed is only theoretical.
  - The trajectory of horizontal throw with this speed would copy exactly the surface.

$$v_I = \sqrt{\frac{\kappa M}{R}} = \sqrt{R\, a_g} = 7.6\, km s^{-1}$$

Work in the gravitational field – potential energy I.

- Let us calculate the work that has to be done to move mass point m in the gravitational field of mass M should it be moved from point A to point B.
- Since the attractive force depends only on the distance it is distance what matters so we move from r_A to r_B

$$W = m\int_{r_A}^{r_B} \frac{\kappa M}{r^2}\,dr = -\kappa m M\left(\frac{1}{r_B} - \frac{1}{r_A}\right) = \Delta E_p$$

Work done by external agent can be identified with the change of potential energy

Work in the gravitational field – potential energy II.

- Since the work is the difference of potential energy any convenient calibration constant can be added the potential energy.

$$E_P(r) = -\frac{\kappa m M}{r} + c$$

- Often this calibration is used: $c=0$ so $E_P=0$ in the infinity
- Near the surface of the Earth $E_P=0$ in some convenient point, where $h=0$. Then

$$E_P(h) = mgh$$

Potential of the Gravitational field φ I

Potential is potential energy of a unit mass

$$\varphi(r) = \frac{E_P(r)}{m} = -\frac{\kappa M}{r} + c$$

Calibration when $c=0$ is sometimes called the absolute calibration.

Conservative Fields

- Gravitation field is an example of a conservative field. Another common conservative field in the electrostatic field. Their definition is:
  - The total work needed to move a body along any closed trajectory is zero.
  - From this it follows that the work necessary to move a body from a point A to a point B doesn't depend on the path but it must depend on some scalar property in these points. This property is the potential energy.

$$W(A\text{->}B) = E_P(B) - E_P(A)$$

Potential of the Gravitational field φ II

- $W(A\text{->}B) = E_P(B) - E_P(A) = m\varphi(B) - m\varphi(A)$
- It is necessary to understand the difference between potential, which is the property of the field at some point, and potential energy, which is the property of a mass body at this particular point of the field.
- Since potential is the property of the field it can be used to describe the field alternatively to the intensity. This has several advantages :
  - it is a scalar function
  - the superposition principle leads to simple arithmetic operations

The relation of potential and potential energy

- If the potential or potential energy can be used alternatively to describe the field we have to know relations between them:

$$\varphi(\vec{r}) = \frac{W}{m} = \frac{1}{m}\int_{r_B}^{r_A} -\vec{F}\cdot d\vec{r} = \int_{r_B}^{r_A} -\vec{E}\cdot d\vec{r}$$

$$d\varphi(\vec{r}) = -\vec{E}\cdot d\vec{r}$$

$$\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = -\vec{E}\cdot d\vec{r}$$

$$grad\varphi\cdot d\vec{r} = -\vec{E}\cdot d\vec{r}$$

$$grad\varphi = -\vec{E}$$

$$\vec{E}(\vec{r}) = -\frac{d\varphi(\vec{r})}{d\vec{r}}$$

The Gradient

- The gradient of a scalar function is a vector that has
  1. Direction of the fastest growth of the function in the given point
  2. Magnitude given by the increase of the function in unit distance in this direction from the given point. :

$$\left(\frac{\partial\varphi}{\partial x};\frac{\partial\varphi}{\partial y};\frac{\partial\varphi}{\partial z}\right)=grad\left(\varphi(\vec{r})\right)$$

- The gradient is the 3D version of the differential :

$$d\varphi\left(\vec{r}+d\vec{l}\right)=d\varphi(\vec{r})+d\vec{l}\cdot grad\left(\varphi(\vec{r})\right)$$

- The properties of the gradient stem from the fact that the dot product is maximal when the factors $d\vec{l}$ and $grad(\varphi(\vec{r}))$ are parallel.

Example of 2D Calculation of a Gradient

Let's have a function of two variables: $h(x,y) = 50e^{-x^2-4y^2}$

What is the change of *h* in the point (1, 2) in the direction toward (4,3)?

$$\vec{u}^{\,0} = \frac{\vec{u}}{|\vec{u}|} = \frac{(4,3)-(1,2)}{\sqrt{10}} = \left(\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{10}}\right)$$

$$\left(\frac{\partial h}{\partial x};\frac{\partial h}{\partial y};\frac{\partial h}{\partial z}\right) = grad(h(\vec{r}))$$

$$\frac{\partial h}{\partial x} \equiv h_x(x,y) = -100xe^{-x^2-4y^2} \Rightarrow h_x(1,2) = -100e^{-17}$$

$$\frac{\partial h}{\partial y} \equiv h_y(x,y) = -400ye^{-x^2-4y^2} \Rightarrow h_y(1,2) = -800e^{-17}$$

$$Dh = grad(h(1,2)) \bullet \vec{u}^{\,0} = \frac{-1100e^{-17}}{\sqrt{10}}$$

Why had the Columbia crushed? Conserving the total energy

$$\Delta W = \Delta E = \Delta E_k + \Delta E_p$$

If there is no work done on the system its the total energy is conserved.

The total energy of the satellite :

$$E = \frac{mv^2}{2} - \frac{\kappa m M}{r} = -\frac{\kappa m M}{2r} = konst\ .$$

When the satellite needs to approach a planet (to land) it has to decrease its potential energy using motors but until it reaches the resistance of the upper atmosphere, as its height decreases its speed increases. In some period of the flight, until the satellite can fly as an airplane it has to lower its kinetic energy and its surface has to withstand extreme temperatures. It must be covered by a layer of special materials which must not be damaged…

The scalar ≡ dot product

$$Let\ c = \vec{a} \cdot \vec{b}$$

Definition I. (components)

$$c = \sum_{i=1}^{3} a_i b_i$$

Definition II. (projection)

$$c = |\vec{a}|\left|\vec{b}\right| \cos\varphi$$

Can you proof their equivalence?

The vector or cross product I

Let ***c*** *=* ***a*** *.* ***b***

Definition (components)

$$c_i = \varepsilon_{ijk} a_j b_k$$

The magnitude |***c***|

$$|\vec{c}| = |\vec{a}|\left|\vec{b}\right| \sin\varphi$$

Is the surface of a parallelepiped made by ***a***,***b***.

The vector or cross product II

The vector c is perpendicular to the plane made by the vectors ***a*** and ***b*** and they have to form a right-turning system.

$$\vec{c} = \begin{Vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{Vmatrix}$$

$\varepsilon_{ijk} = \{1$ (even permutation), -1 (odd), 0 (eq.)$\}$

Physics

6

Elasticity and Fracture

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Main Topics

- Introduction
- Atomic hypotheses
  - Long range forces between atoms and molecules
- Introduction into elasticity and fracture
  - Stress
  - Strain
  - Stress/strain diagram
  - Hook's law
  - Perpendicular deformation, Poisson's constant
  - Tensors of stress and strain

Introduction

- On our path to describe the Nature we started from the simplest object a mass point, then we proceeded to a general system of mass points and then to rigid bodies. At each of these steps we revealed new properties of matter.
- However, every kid who performed an experiment pulling a chewing gum out of his mouth knows that bodies can be deformed and can even break.
- The description of these effects is complicated and even at its easiest level needs some knowledge on the micro-world interactions.

Atomic hypotheses I

- Richard Feynman – one of the greatest physicists of the 20th century and the author of famous textbook 'Feynman Lectures on Physics' says that if we could leave only one sentence to the following generations it should be: "All things are made of atoms – little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another".
- The most convenient unit to measure atoms and bonds between them is the non-SI unit angström $1Å = 10^{-10}$ m

Atomic hypotheses II

- If we enlarge an apple about 10^8 times to the size of the Earth the atoms would have the size of apples.

$$6 \cdot 10^{-2} \cdot 10^8\ \mathrm{m} = 6 \cdot 10^6\ \mathrm{m}$$
$$\sim 6\ \mathrm{\acute{\AA}} \cdot 10^8 = \sim 6 \cdot 10^{-10} \cdot 10^8\ \mathrm{m} = \sim 6\ \mathrm{cm}$$

It interesting that even at this magnification

- the atomic core would be too small to be visible by eyes
- the surface of the Earth would be 60 times too small to not reach the closest star

- In the vicinity of the astronomical observatory in HK there is a planetary model of the Sun's planetary system $1:10^9$.

Long range forces I
Cherchez le puits (de potential)

- Building blocks of matter –atoms or molecules act mutually by long-range forces which have the following properties:
  - at macroscopic distances they are negligible
  - at shorter distances they are attractive
  - at even shorter distances they become repulsive
  - at least one equilibrium distance exists where the attractive and repulsive forces compensate

Long range forces II

- Acting of long-range forces in micro-world can be illustrated by a simplified potential well:
  - With its help many properties can be explained e.g.:
    - the existence of condensed matter
    - elastic behavior of matter
    - thermal expansion
  - near its minimum the well can be approximated by a parabola

*Long range forces III

- Interaction of molecules is often modeled by Lenard-Jones potential 6-12

$$\varphi(r;\varepsilon,r_0)=\varepsilon[(\tfrac{r}{r_0})^{-12}-2(\tfrac{r}{r_0})^{-6}]$$

  - ε is the depth of the potential well
  - r_0 is the equilibrium distance
  - repulsive forces should be exponential but this model enables easier evaluation of interaction integrals with acceptable accuracy

*Long range forces VI

- Interaction of atoms is often modeled by Morse potential

$$\varphi(r;a,\varepsilon,r_0) = \varepsilon[\exp a(r_0 - r) - 2]\exp a(r_0 - r)$$

  - ε is the depth of the potential well
  - r_0 is the equilibrium distance
  - a limits the reach of the interaction
  - Enables to find easily the stationary states and the application of anharmonic interaction an improvement reflecting the fact that matter can be easier stretched than compressed

Elasticity I

- After the description of the long-range forces acting between constituent particles of matter, it is clear that the model of a rigid body, that is in many cases a reasonable approximation, strictly doesn't work.
- If external force acts on a body, its shape changes so it corresponds the equilibrium of external and internal forces.
- The response to a change of the external forces is also a change of the internal forces which try to oppose them. The result is a new equilibrium corresponding to the current state of stress..

Elasticity II

- Our potential well model works for large number of materials. But it is an oversimplification so materials with very exotic properties do exist.
- We can nevertheless accept that very small deformations are elastic, which means that after disappearance of the external force the body returns to its original shape.
- To make the description as simple as possible we introduce convenient quantities :

Mechanical stress I

- Experiment shows that for the deformation effect the force has to be related to the surface on which it acts. The appropriate quality is mechanical stress, shortly stress

$$\vec{\sigma} = \frac{d\vec{F}}{dS} = \frac{\Delta\vec{F}}{\Delta S}$$

- The SI unit of mechanical stress is *1 Pascal [Pa]=Nm*$^{-2}$. Hydrostatic pressure is a special case of stress.

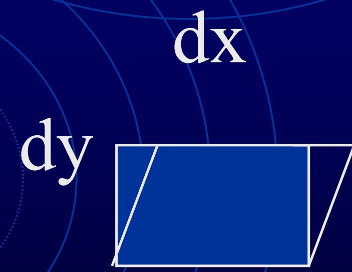
Mechanical stress II

- The response of materials to external force may be complicated but even for the most simple ones (highly symmetric, homogeneous and isotropic) it is different at least in the normal and tangent direction. Therefore, it makes sense to decompose generally acting stress to normal and tangent components :

$$\sigma_n = \frac{dF_n}{dS} \qquad \sigma_t = \tau = \frac{dF_t}{dS}$$

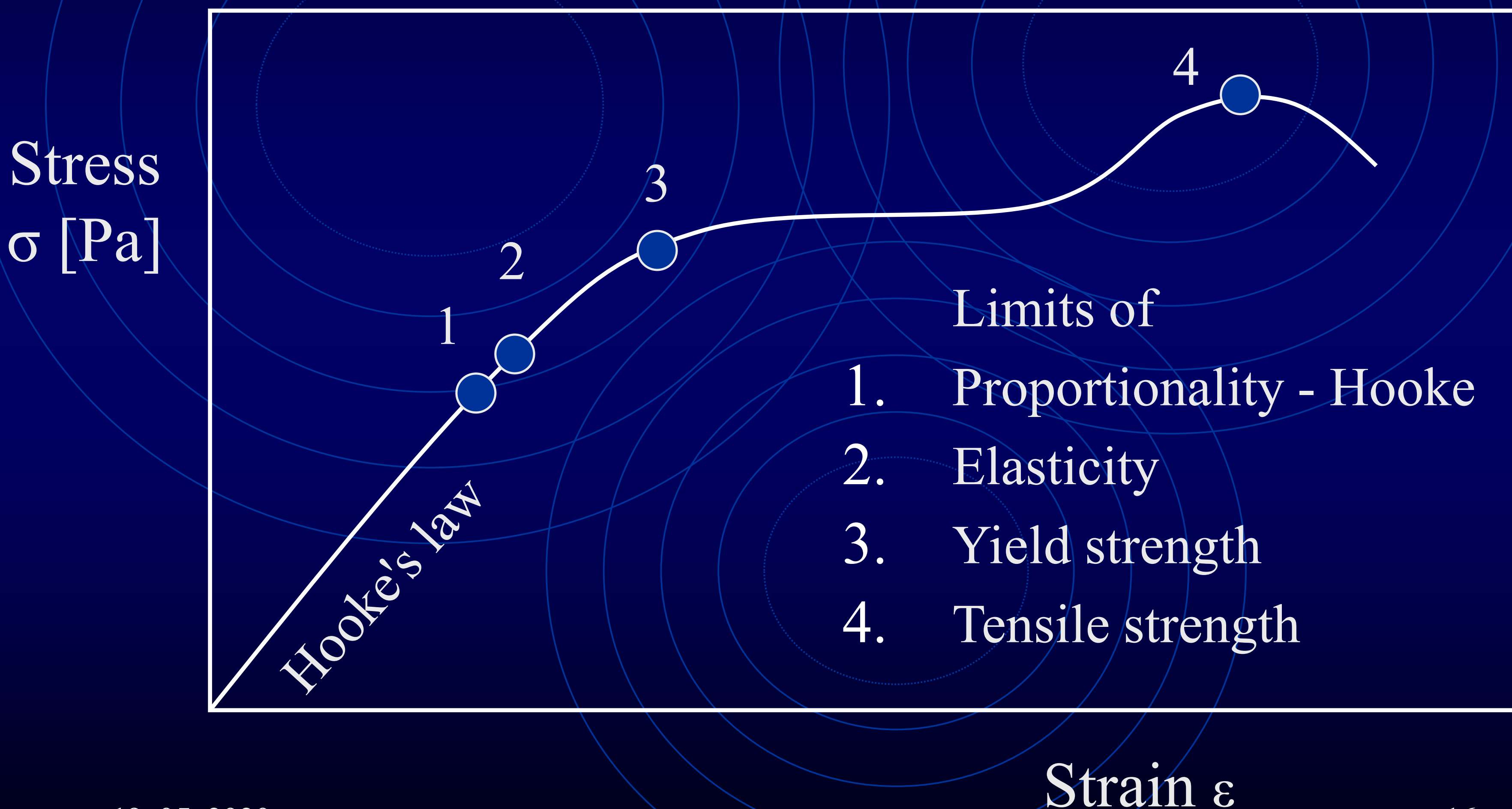
Deformation - Strain

- The response of materials to external force is always proportional to the undeformed dimension. Fro this reason it is convenient to define strain as relative deformation. So we define relative
  - prolongation
  - $$\varepsilon = \frac{dl}{l_0}$$
  - shear deformation
    $$\gamma = \frac{dx}{dy}$$



  - compression
    $$-\frac{dV}{V_0}$$

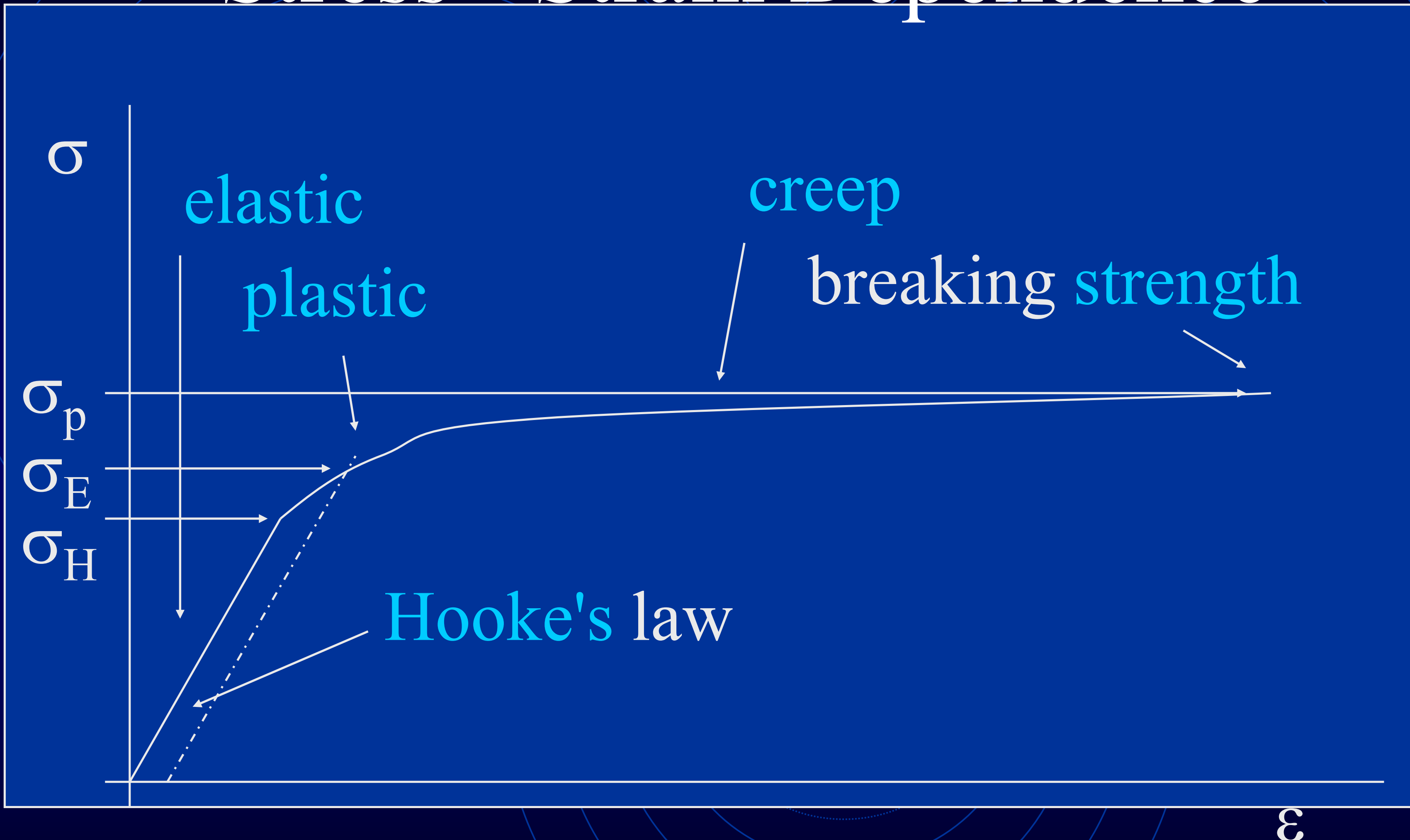
Stress - Strain Diagram



Stress - Strain Dependence

- The stress - strain dependence is usually depicted as this diagram. It has the following regions and limits:
  - proportionality ... here Hooke's law holds $< \sigma_H$
  - elasticity ... returns to original shape $< \sigma_E$
  - plasticity ... rest deformation remains
  - yield ... considerable change of behavior
  - strength ... fracture of material $< \sigma_P$

Stress - Strain Dependence



Hooke's law I

- For very small (literally infinitely small) deformations this holds :

$$\sigma = E\varepsilon$$

$$\tau = G\gamma$$

$$p = -K\frac{dV}{V}$$

Quantities *E*, *G* and *K* are Young's modules, they are a measure the ability to resist the particular deformation and usually are large $\sim 10^{10}$ Pa.

Hooke's law II

- The moduli are called :
  - E ... Young's modulus in prolongation
  - G … Young's modulus in shear
  - K … modulus of volume elasticity
- Often their reciprocal values are used. They are called compliances and are typically small.

Hooke's law III

- Longitudinal deformation is accompanied by transversal deformation
- E.g. longitudinal prolongation *Δl* of a stick l_0 is always accompanied by shortening *Δa* of any transversal dimension *a* : $l_0 \cdot a \to \left(l_0 + \Delta l\right) \cdot \left(a - \Delta a\right)$
- In Hooke's region the relative transversal shortening η is proportional to the longitudinal stress and also strain. :

$$\eta = \frac{-\Delta a}{a} = \frac{a - a'}{a} = k_1 \sigma_n$$

Hooke's law IV

- The transversal change is characterized by another material parameter ν or m :

$$\eta = k_1 \sigma_n = k_1 E \varepsilon = \nu \varepsilon = \frac{1}{m} \varepsilon = \frac{1}{mE} \sigma_n$$

- Poisson constant : $m = \varepsilon / \eta$
- Poisson number (ratio): $\nu = 1/m = \eta / \varepsilon$

  Large poisson number means relatively large transversal deformation

Hooke's law V

- Longitudinal, transversal and shear deformations are not independent :

$$G = \frac{mE}{2(m+1)} = \frac{E}{2(1+\nu)}$$

  where m is Poisson's constant and ν Poisson's number, defined above

- For the quotient of volume compressibility it holds :

$$\gamma \equiv -\frac{\Delta V}{V}\frac{1}{p} = \frac{3(m-2)}{mE} = \frac{3(1-2\nu)}{E} = \frac{1}{K}$$

Deformation of un-isotropic materials I

- In general case both stress and strain have to be expressed as symmetric tensors of the second order $\boldsymbol{\tau}$ and $\boldsymbol{\varepsilon}$.
  - τ_{ij} is j-th component of stress acting on the surface element perpendicular to the axis i.
  - ε_{pq} is deformation of the surface element perpendicular to the axis p in the direction of axis q.

Deformation of un-isotropic materials II

- Then the generalized Hooke's law is written as :

$$\tau_{ij} = C_{ijpq}\ \varepsilon_{pq}$$

  - C_{ijpq} in the *36* independent elastic coefficients.
  - Any symmetry in the material means also symmetry in ***C*** and decrease of the number of independent material parameters.
  - The most trivial symmetry is in exchange the couples *ij* and *pq*. This decreases the number of independent parameters to *21*. This corresponds to the least symmetric triclinic system.
  - Amorphous or symmetric polycrystallic materials behave as isotropic and have only two elastic parameters *E* nd *G*.

The scalar ≡ dot product

$$Let\ \ c = \vec{a} \cdot \vec{b}$$

Definition I. (components)

$$c = \sum_{i=1}^{3} a_i b_i$$

Definition II. (projection)

$$c = |\vec{a}|\left|\vec{b}\right| \cos\varphi$$

Can you proof their equivalence?

Physics

7

Mechanics of Fluids

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Main Topics

- Introduction into mechanics of fluids
- Hydrostatics (fluids at rest)
  - Pascal's principle
  - Buoyancy and Archimedes' law
- Hydrodynamics (fluids in motion)
  - Conservation laws
    - Equation of continuity
    - Conservation of momentum
    - Bernoulli's law
    - Bernoulli's principle – hydrodynamic paradox
  - Newton fluids – viscosity

Intro into mechanics of fluids I

- Fluids is a common name for liquids and gases.
  - In common they have almost zero shear modulus. So they easily accept shape of container they are confined in.
  - Liquids are almost incompressible – they have almost infinite compression modulus.
  - They can be separated relatively more easily.
  - In most cases the effects due to atomic structure are negligible so fluids can be considered to continuous matter, called continuum.

Intro into mechanics of fluids II

- Looking at elastic parameters fluids are defined the following way :
  - Liquids … K very big, G small
  - Gases ... K given by the EOS, G small
- We start with simplification the ideal liquid with $K=infinity$, $G=0$

  which are incompressible and flow ideally

Hydrostatics of ideal liquids I

- Hydrostatics deals with fluids at rest = in equilibrium regardless how long it took to reach it.
- Although the prefix 'hydro' indicates dealing with water the traditional name hydrostatics covers study of any fluid at rest.
- We consider ideal liquid moreover homogeneous and isotropic .
- It is convenient to characterize fluids by densities which are physical quantities related to unit volume.

Hydrostatics of ideal liquids II

- The most common and important are :
  - density ρ is the mass of unit volume:

    $$\rho = m/V,\ [\rho] = kg\ m^{-3}$$

  - density of acting forces $\vec{f}$ that is the force per unit volume

    objemu : $\vec{f} = \dfrac{\vec{F}}{dV}$, $[f] = N\ m^{-3}$

  - pressure can be considered as density of pressure energy : $[p] = N/m^2 = J/m^3$

Elementary equation of hydrostatics I

- The stress tensor for the ideal liquid can be simplified employing the Pascal's principle :

$$\tau_{ij} = -p\delta_{ij}.$$

  δ_{ij} is Cronecker's delta that can reach two values : $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$.
- $p = F/S$ [Pa] is the pressure – normal stress.
- It can be shown that the elementary equation for the equilibrium of an element of continuum holds:

$$\frac{\partial \tau_{ij}}{\partial x_i} + f_j = 0$$

Elementary eq. of hydrostatics II

- After inserting the stress tensor :

$$-\frac{\partial p}{\partial x_j} + f_j = 0 \Leftrightarrow grad\ p = \vec{f}$$

- The force acts in the direction of the largest increase of pressure or the largest increase of pressure is in the direction of the acting force.
- Especially, if the force is caused by a conservative field which has potential, such as gravitation :

$$\vec{f} = -\rho\, grad\ \varphi$$

Elementary eq. of hydrostatics III

- We obtain : $grad\ p = -\rho\, grad\,\varphi$
- And finally after integration : $dp = -\rho d\varphi$
- From this equation it can be seen that the points with the same pressure lay on the equipotential surfaces and pressure increases with the decrease of potential as well as with the increase of density.

Elementary eq. of hydrostatics IV

- All interfaces of fluids, including their level, which is the interface between liquid and atmosphere, are equipotential surfaces. Therefore :
  - Water level is not strictly horizontal :
  - It bends near the walls of the dish
  - It copies the surface of the Earth but it is influenced by the its rotation, minor changes of potential due to inhomogenities of mass but also tidal effects – the common influence of the Moon and the Sun
  - In non-inertial system, e.g. in rotating dish, it is perpendicular to the resulting force at any point.

Pascal's principle – Pressure I

- Since in any point the tangent component of stress is always zero only the normal stress component = pressure is present and it is isotropic :
  - The measured pressure at any point of submerged little sphere is isotropic or the sphere would move by itself.
  - Explanation of any hydrostatic property of fluids or its use is based on this. E.g. Hydraulics :

  If we connect two cylinders with pistons of different size and can neglect pressure caused by the liquid itself then pressure is equal anywhere in the system and on the pistons the forces are proportional to the cross section of the pistons

$$F_1/S_1 = p_1 = p_2 = F_2/S_2$$

Pressure near the Earth surface

- Near to the Earth surface
  - the gravitation potential at some height *z* above some zero reference point is

$$\varphi = gz$$

  - the *z* axis is vertical and grows upwards
  - If we allow the dependence of density on z (compressibility) then : $\dfrac{dp}{dz} = -\rho(z)g$

Pressure of incompressible liquid near the Earth surface behaves linearly

- The density of incompressible liquids is constant :

$$dp = -\rho g dz$$

- The integration leads to linear decrease of pressure with height : $p(z) = p_0 - \rho g z$
- For practical reasons e.g. by divers the same relation expressed as linear growth of pressure with depth from the level : $p(h) = b_0 + \rho g h$

Pressure of ideal gas near the Earth surface behaves exponentially

- Let's have isothermic column of ideal gas obeying the i.g. EOS or Boyle-Marriot law :

$$pV = p_0 V_0 \Rightarrow \frac{m}{pV} = \frac{m}{p_0 V_0} \Rightarrow \rho = p\frac{\rho_0}{p_0}$$

- Then : $$\frac{dp}{dz} = -\frac{\rho_0}{p_0} pg$$

- This differential equation of the first order can be solved by integration after separation of variables :

$$p(z) = p_0 \exp(-\frac{\rho_0 g z}{p_0})$$

Buoyancy, Archimedes' Law I

- The body immersed into fluid is lifted by a buoyant force equal to the weight of the fluid displaced by the body.
- Buoyant force is the result of forces exerted by pressure of the fluid which wants to get 'back' to volume from which it was displaced and where it wants at least theoretically* to return.
- Since pressure growths with depth the resulting force points upwards.
- *Situations when a body is immersed or liquid is poured over the body conceptually differs because in the former case the liquid really used to be in the volume where the immersed body is now while in the latter case this is not so. Archimedes' law holds in both cases anyway!

Archimedes' Law II

- Archimedes' law
  - is connected to the growth of pressure with depth
  - can be derived using a body of a special shape
  - or more generally as an equilibrium of volume and surface forces. This doesn't need constant density or incompressibility and works even through several interfaces of non-miscible fluids.

Archimedes' Law III

- Let us have a cylinder of height h and cross section S immersed in ideal liquid of density ρ_0 :
  - Pressure forces acting on cylinder jacket cancel.
  - Only forces on the top and bottom surface don't compensate and the resulting force is :

$$F = Sh\rho_0 g.$$

  - But this is exactly equal to the weight of the displaced liquid.

Archimedes' Law IV

- In liquid at rest i.e. in equilibrium let's consider a body of the same liquid of any shape, perhaps separated from the rest by imaginary membrane.
  - This body has mass and its weight points down.
  - On each surface element of this body some pressure acts causing a 'pressure force'.
  - Since the body is in equilibrium the sum of these pressure forces must exactly compensate the weight : it points up and has the same magnitude.
  - If a body of the same shape was filled with material with higher or lower density the buoyant force remains and the weight would be either higher or lower.

Introduction in hydrodynamics

- Hydrodynamics belongs among the most complex fields of classical physics.
- The name hydrodynamics is used more generally for fluid dynamics of any fluid.
- Using the conservation laws some simple conclusions can be derived for slow flow of ideal liquids.
- Later we include viscosity and deal with simply behaving Newton's liquids.

Laminar flow

- Fluids in laminar motion can be described using:
  - Trajectories, on which particles move in time. By a particle we mean some very small yet microscopically large volume.
  - Streamlines, curves tangent to the velocity of flow in any point. Streamlines may form a stream pipe through the walls of which the fluid can't flow and the walls are made of the same fluid.

Conservation laws

- For the flow of ideal liquids we can employ conservation of :
  - Quantity of matter – equation of continuity
  - Linear momentum
  - Energy – Bernoulli's equation

Equation of continuity

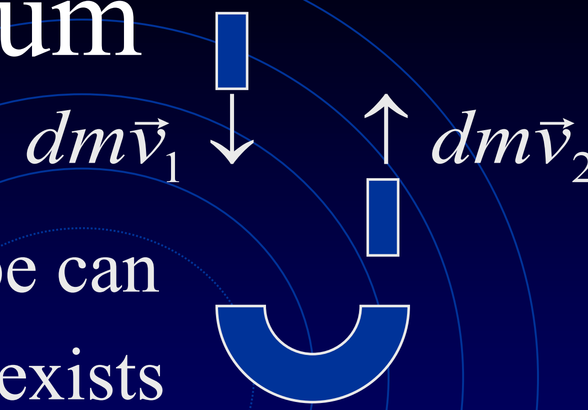
- Consider a stream pipe of any shape. If the liquid is incompressible and doesn't accumulate anywhere the volume that flows through any cross section per unit time i.e. the volume rate is equal.
- Lets consider two cross sections S_1 with the velocity is v_1 and cross section S_2 and velocity v_2 :

$$S_1 v_1 = Q_1 = Q_2 = S_2 v_2$$

- For compressible liquids the mass rate is constant :

$$S_1 v_1 \rho_1 = S_2 v_2 \rho_2$$

Conservation of l. momentum



- A volume element in a streamline pipe can change its direction only if an impulse exists allowing the corresponding change of linear momentum:

$$\vec{F}dt = \Delta\vec{p} = (\vec{v}_2 - \vec{v}_1)dm = (\vec{v}_2 - \vec{v}_1)Q\rho dt \Rightarrow$$

$$\vec{F} = (\vec{v}_2 - \vec{v}_1)Q\rho$$

- Above that surroundings of the streamline pipe must support also the difference of pressure forces.
- Changing flow rate in a hose leads to new equilibrium.

Conservation of energy

- Bernoulli's equation expresses the conservation of (the density) of energy :

$$\frac{\rho v^2}{2} + \rho g h + p = \frac{E}{V} = konst.$$

- It is often used in forms with quantities of different dimensions, for instance, length :

$$\frac{v^2}{2g} + h + \frac{p}{\rho g} = konst.$$

Conservation of energy Bernoulli's equation I

- Lets consider a piece of a streamline pipe the boundary surfaces of which can be described by speed v_i, pressure p_i, height h_i and cross section S_i.
- By acting of pressure forces a volume ΔV moves from the second boundary to the first within some time period Δt .
- The pressure forces acting on the boundaries are

$$F_i = S_i \, p_i.$$

- Work, done by these forces (over Δt) is equal to the increase of the total energy of the volume ΔV.

Bernoulli's equation II

- Therefore :

$$\Delta t(F_1 v_1 - F_2 v_2) = E_{k2} + E_{p2} - (E_{k1} + E_{p1})$$

- After substitution if the energies :

$$\Delta t(p_1 S_1 v_1 - p_2 S_2 v_2) = \frac{\Delta m(v_2^2 - v_1^2)}{2} + \Delta m g(h_2 - h_1)$$

- And employing the equation of continuity :

$$\Delta V(p_1 - p_2) = \frac{\Delta m(v_2^2 - v_1^2)}{2} + \Delta m g(h_2 - h_1)$$

Bernoulli's equation III

- Finally after rearranging and dividing by ΔV (which relates the terms to the unit volume) :

$$p_1 + \frac{\rho v_1^2}{2} + \rho g h_1 = p_2 + \frac{\rho v_2^2}{2} + \rho g h_2$$

- Daniel Bernoulli 1700-1783, Swiss
- The total energy of fluid in motion has three components : pressure, kinetic and potential.

The use of Bernoulli's E. Flow through a small hole I

- Bernoulli's equation can be used to solve many practical problems.
- Often some of its six terms cancel or can be neglected which simplifies the solution.
- Consider a liquid in wide dish with a small hole in the depth *h* under the level.
- In general Bernoulli's equation we can do several simplifications :

The use of Bernoulli's E. Flow through a small hole II

$$p_1 + \frac{\rho v_1^2}{2} + \rho g z_1 = p_2 + \frac{\rho v_2^2}{2} + \rho g z_2$$

- Both pressures are atmospheric : $p_1 = p_2$.
- The depth is the difference of vertical coordinates : $h = z_1 - z_2$
- The velocity v_1 can be neglected.

- After cancelling ρ and rearranging : $v_2 = \sqrt{2gh}$

  This is Torricheli formula and had been known about a century before Bernoulli..

The use of Bernoulli's E. Flow through a small hole III

- If the speed v_1 can't be neglected we use the equation of continuity $v_1 = v_2 S_2/S_1$:

$$\frac{\rho v_2^2 S_2^2}{2S_1^2} + \rho g z_1 = \frac{\rho v_2^2}{2} + \rho g z_2$$

- From a little more complicated BE we get :

$$v_2 = S_1 \sqrt{\frac{2gh}{S_1^2 - S_2^2}}$$

(this has naturally meaning only if $S_1 > S_2$)

The use of Bernoulli's E. hydrodynamic paradox

- Considering two points with the same height from the Bernoulli's equation we easily see that in the point where the velocity is higher the pressure must be lower = Bernoulli's principle
- Many effects and inventions are based on this effect : from slamming of doors in draught to flying of airplanes.
- It is mainly useful to measure speed of the flow.

The use of Bernoulli's E. Pitot's pipe

- Lets immerge two pipes into the moving liquid so that their ends are in the same height. The end of the first pipe is parallel to the flow the end of the other is normal so that $v_2 = 0$.
- In the i^th pipe the liquid reaches the height z_i, corresponding to the pressure $p_i = \rho g z_i$ at its end

$$\rho g z_1 + \frac{\rho v_1^2}{2} = \rho g z_2 \Rightarrow v_1 = \sqrt{2g(z_2 - z_1)}$$

- it contains only the difference of the heights z_i.
- it is used to measure of fluid in a pipe

The use of Bernoulli's E. Venturi's pipe I

- Venturi's pipe (funnel) :
  - a funnel shape pipe is immersed into the horizontal stream. Perpendicularly two pipes are connected one in a place with cross section S_1, the other S_2.
  - In the i^{th} pipe the liquid reaches the height z_i, corresponding to the pressure $p_i = \rho g z_i$ at its end

$$\rho g z_1 + \frac{\rho v_1^2}{2} = \rho g z_2 + \frac{\rho v_2^2}{2} \wedge S_1 v_1 = S_2 v_2$$

The use of Bernoulli's E. Venturi's pipe II

- From both equations :

$$\rho g(z_1 - z_2) + \frac{\rho v_1^2}{2} = \frac{\rho v_1^2 S_1^2}{2S_2^2}$$

- For the speed v_1 and volume rate Q we get :

$$v_1 = S_2 \sqrt{\frac{2g(z_1 - z_2)}{S_1^2 - S_2^2}}$$

$$Q = S_1 v_1 = S_1 S_2 \sqrt{\frac{2g(z_1 - z_2)}{S_1^2 - S_2^2}}$$

Viscous Liquids
Newton's Liquids I

- In motion neighboring layers influence each other by shear stress which depends on the mutual speed and viscosity of the liquid.
- Consider horizontal flow in the direction of x-axis. Then according the Newton's law the shear stress acting in the direction of the flow is :

$$\sigma_x = \tau = \eta D \equiv \eta \frac{dv}{dy} = \eta \frac{dx}{dydt} = \eta \frac{d\gamma}{dt} \equiv \eta \dot{\gamma}$$

Newton's Liquids II

- dynamic viscosity η – measure of the resistance to the flow $[\eta] = kg\ m^{-1}s^{-1} = Nm^{-2}s = Pa\ s$
- older unit is Poise $[P]=gcm^{-1}s^{-1}=0.1\ Pa\ s$
- The reciprocal quantity is tekutost:

  $\varphi = 1/\eta$
- often viscosity is related to the density is used; kinematic viscosity (nju) $\nu = \eta/\rho$
- D – the perpendicular gradient of speed is equal to the time derivative (change) of the shear strain.

Newton's Liquids III

- For better illustration of meaning of viscosity lets consider cylindrical dish with cylindrical stirrer and Newton's law in the form : $$\tau = \eta \frac{dv}{dy}$$
- Should the stirrer rotate always with the same frequency then higher shear moment and thereby higher power of the motor is needed for liquid with higher viscosity.
- The same is if for one particular liquid we need to increase the frequency of stirring.

Newton's Liquids IV

- Dynamic and kinematic viscosity of some selected liquids :

	η (eta) [Pa s]	ν (nju) [m^2/s]
ETOH	$1.2\ 10^{-3}$	$1.51\ 10^{-6}$
mercury	$1.5\ 10^{-3}$	$1.16\ 10^{-7}$
petrol	$2.9\ 10^{-4}$	$4.27\ 10^{-7}$
oil	$2.6\ 10^{-2}$	$2.79\ 10^{-4}$
water	$1.005\ 10^{-3}$	$8.04\ 10^{-7}$

Newton's Liquids V

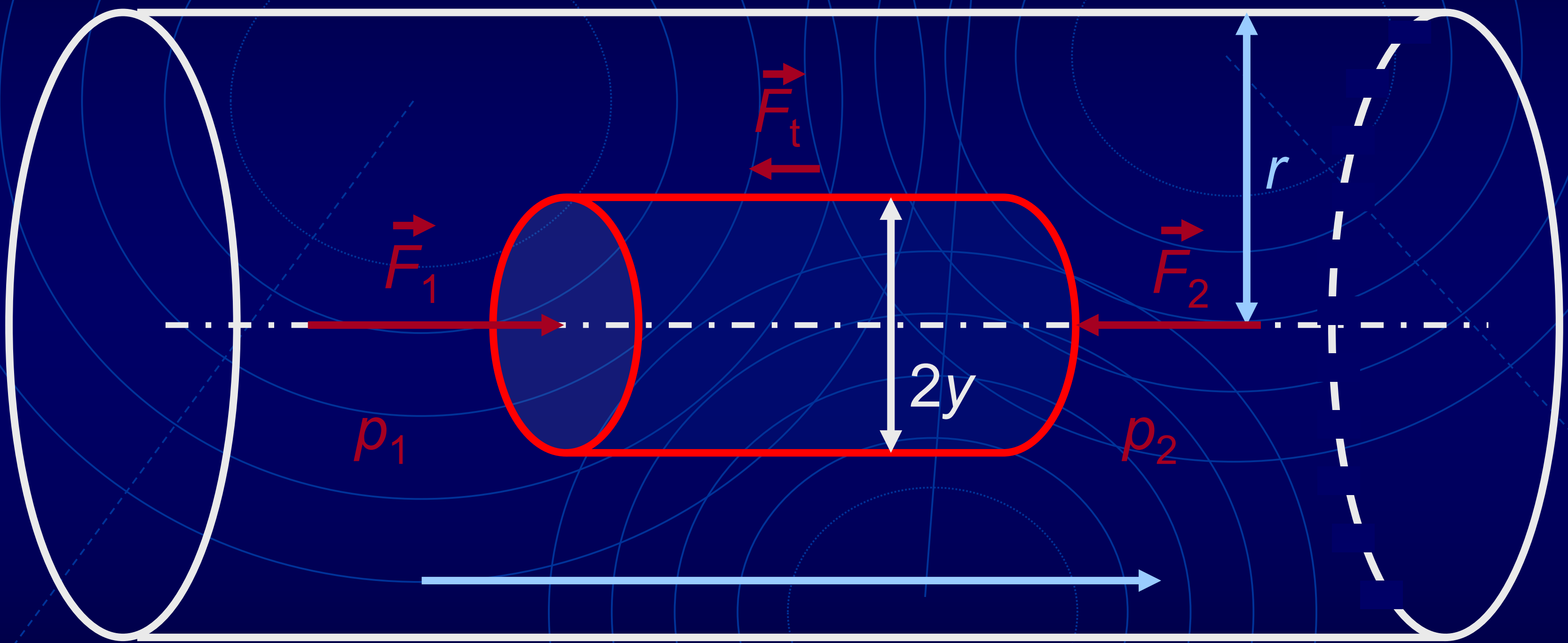
- Viscosity :
  - decreases volume rate of liquid at given conditions
  - causes velocity distribution in the cross section of a pipe, close to zero near the pipe surface and maximum in the middle.
- We show that in the pipe of circular cross section the velocity distribution is parabolic.

Newton's Liquids VI

- Consider a cylinder with radius *y* coaxial with the pipe in laminar floating liquid :
  - pressure acts on its caps ($p_1 > 0$, $p_2 < 0$)
  - friction of the surrounding layers acts on its jacket.
  - should this cylinder move in uniform motion all the forces must be in equilibrium :

$$\pi y^2 (p_1 - p_2) + 2\pi y \,\Delta l \,\eta \frac{dv}{dy} = 0$$

Calculation of the volume rate of viscous liquid



Direction of the motion of liquid

Newton's Liquids VIII

- Suppose that $p_1 > p_2$ so the liquid flows in the direction of growth of x-coordinate.
- The sign + would mean that the friction force would have the direction of the velocity.
- Since the first term is positive the friction force must be negative and the velocity decreases in the direction from the axis of symmetry.

Newton's Liquids IX

- After substitution $\Delta p = p_1 - p_2$:

$$dv = -\frac{1}{2\eta}\frac{\Delta p}{\Delta l}\, y\, dy$$

- And integration :

$$v(y) = -\frac{1}{4\eta}\frac{\Delta p}{\Delta l} y^2 + k$$

Newton's Liquids X

- If the pipe has radius *r* we get the value of the integration constant *k* from the boundary condition *v(r) = 0* :

$$k = \frac{1}{4\eta}\frac{\Delta p}{\Delta l} r^2$$

- and we get parabolic velocity profile :

$$v(y) = \frac{1}{4\eta}\frac{\Delta p}{\Delta l}(r^2 - y^2)$$

Newton's Liquids XI

- Important and easier measurable quantity is the volume rate. The whole cross section we can divide to thin circles with radius y in which the velocity is constant :

$$dQ_v(y) = 2\pi y dy v(y) = \frac{1}{2\eta}\frac{\Delta p}{\Delta l}(r^2 - y^2)\, y\, dy$$

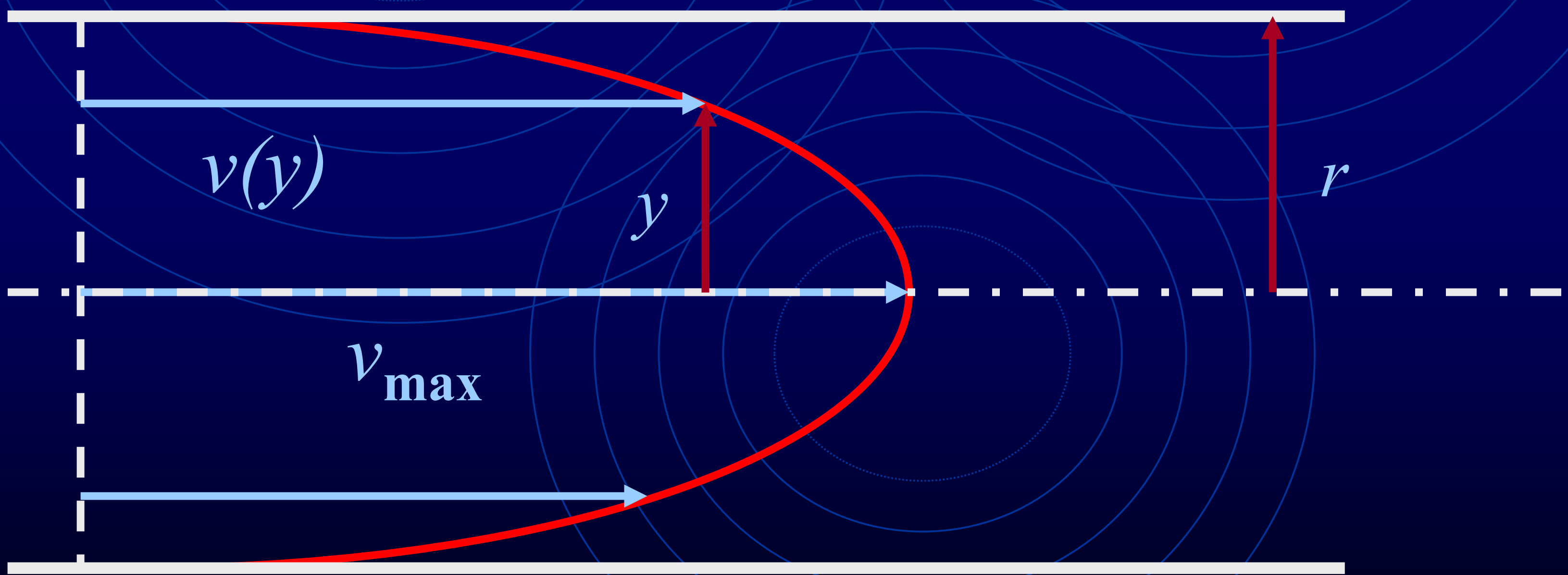
- The total volume rate is obtained by integration:

$$Q_v = \frac{1}{2\eta}\frac{\Delta p}{\Delta l}\int_0^r (r^2 - y^2)\, y\, dy = \frac{\pi r^4}{8\eta}\frac{\Delta p}{\Delta l}$$

- This is the well known Hagen-Poiseuille equation.

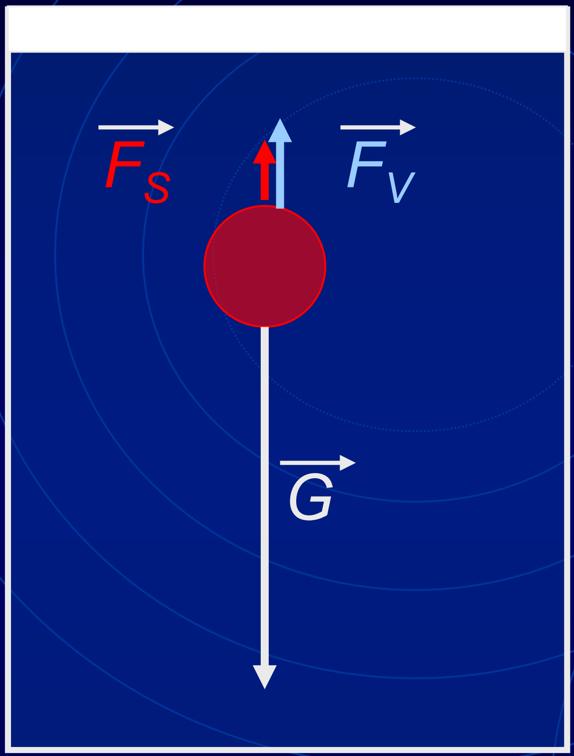
Velocity profile in a pipe of circular cross section

$$v(y;r) = \frac{1}{4\eta}\frac{\Delta p}{\Delta l}(r^2 - y^2) \qquad v_{\max} = \frac{1}{4\eta}\frac{\Delta p}{\Delta l}(r^2)$$



Viscosity measurements – fall of a sphere in liquid

Forces: weight, buoyancy, internal friction



$$G = \frac{4}{3}\pi r^3 \rho g$$

$$F_V = \frac{4}{3}\pi r^3 \rho_0 g$$

$$F_S = 6\pi\eta r v$$

The sphere accelerates until the three acting forces cancel:

$$F_V + F_S = G$$

Newton's Liquids XIV

- Stokes' law :
  - A internal friction force $F = 6\pi r v \eta$ acts on a sphere with radius r moving with small velocity v in liquid with viscosity η.
  - After equilibrium is reached the sphere with density ρ in liquid with density ρ_0 will move with velocity v_t :

$$v_t = \frac{2r^2 g}{9\eta}(\rho - \rho_0)$$

Newton's Liquids XV

- Laminar flow
  - the friction force is proportional to the velocity
  - the velocity is proportional to r^2
  - mean velocity of flow from the H-P equation $\langle v \rangle = Q_v/S$ is also proportional to r^2 and pressure gradient
- Beyond Stokes' law :
  - The friction force is often proportional to v^2 :

    $$F_d = C_d S v^2$$

    where C_d is a parameter depending on the shape

Newton's Liquids XVI

- Reynolds' number is used to estimate whether the flow is still laminar : $R = \dfrac{\rho v r}{\eta} = \dfrac{v r}{\nu}$
  - it holds for a sphere with radius *r*, with the velocity *v*
  - or fluid moving with mean velocity $<v>$ in a pipe of radius *r* :
  - kinematic viscosity is present in the denominator of the second term
- For $R > 1000$ the flow is considered turbulent
- When $R \to 0$ (*r* small and/or ν big) liquids behave very non-standard way.

Physics

8

Oscillations and Waves

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Mains Topics

- Oscillations
  - general, periodic, harmonic, damped, un-damped
- Harmonic oscillations
  - equation of motion and its solutions
  - time dependence of displacement, speed, acceleration, potential and kinetic energy
  - adding oscillations, damped and forced oscillations
- Waves – general and harmonic waves
  - description, periodicity in space and time, transport of energy
  - standing waves, interference, Doppler's effect

Introduction – oscillations and waves I

- Studies of oscillatory or vibration motion have to be added to the previously studied translation and circular movements.
  - their existence stems from the elastic character of interactions between particles of matter
  - they are widely spread in Nature
  - Oscillation energy is an important type of energy

Intro – oscillations and waves II

- Generally oscillation of particle is movement that is limited in space.

  Wave is spreading of vibration movement in space that carries energy but not mass
- Should the mass particle oscillate the following items must exist :
  - equilibrium position where forces don't act on the particle
  - restoring forces – try to return the particle to the e.p.
  - the equilibrium must be stable

Type of oscillations I

- Character of oscillations, the precise time dependence of the displacement is given by :
  - the character of restoring forces - their actual dependence on the displacement. The force field is often conservative and the forces follow from the potential.
  - possible existence of dissipative forces.
- An important subset are periodic oscillations, where the time dependence of displacement repeats periodically.
- An important subset of periodic osc. are harmonic oscillations where the time dependence of displacement can be expressed using a harmonic = goniometric function of time.

Type of oscillations II

- Undamped periodic and also harmonic oscillations can exist only if there are no losses of mechanical energy present.
- If dissipative forces are present the mechanical energy decreases and eventually the movement stops. This is typical for damped oscillations.

Type of oscillations III

- Strictly speaking all real oscillations should be damped.
- Undamped oscillations still are important since:
  - damping can be small or negligible
  - losses of energy can be restored convenient way

Type of oscillations IV

- Harmonic oscillations are very important since:
  - the special restoring forces they need do exist.
  - any periodic oscillation can be expressed as Fourier series of harmonic oscillations
  - any general oscillation can be expressed as Fourier integral of harmonic oscillations
  - Since both addition and integration are linear operations many properties of harmonic oscillations are valid more generally

Simple (undamped) harmonic motion I

- Let's consider the following simple situation :
  - a particle can move only on a straight line which we coincide with the x-axis
  - the equilibrium position is chosen as the origin
  - the restoring force is directly proportional to the displacement

$$F(x) = -kx$$

Simple harmonic motion II

- At first we find work necessary to reach the displacement x moving from the origin :
  - since the restoring force depends on the current displacement χ we have to integrate :

$$E_p(x) = W(x) = \int_0^x k\chi d\chi = \frac{kx^2}{2}$$

  - this result is symmetric: To reach some displacement x either negative of positive we have to supply positive work
  - and it corresponds to parabolic potential well

Simple harmonic motion III

- This type of restoring force can exist in any field e.g. in gravitational or electrostatic which have potential and the potential well can be approximated by parabola at least for small oscillations.
- This restoring force may correspond to Hook's behavior. In this case the proportionality parameter k is proportional to the appropriate Young modulus and it is called the spring constant.

Simple harmonic motion IV

- If we substitute for force from the Newton's 2nd law we get the equation of motion, particularly differential equation of the second order with missing first order :

$$F = m\ddot{x} = -kx$$

- We guess the form of the solution and find its parameters :

$$x(t) = x_0 \sin(\omega t + \varphi)$$

Simple harmonic motion V

- Let's calculate its (first and) second time derivative and substitute it to the original equation : $\dot{x}(t) = x_0\omega\cos(\omega t + \varphi)$

$$\ddot{x}(t) = -\omega^2 x_0 \sin(\omega t + \varphi) = -\omega^2 x(t)$$

$$-m\omega^2 x(t) = -kx(t)$$

- From which we get : $\omega = \sqrt{\dfrac{k}{m}}$

Simple harmonic motion VI

- the time dependence of the displacement is :

$$x(t) = x_0 \sin(\omega t + \varphi) = x_0 \sin(\sqrt{\frac{k}{m}}t + \varphi)$$

  - angular frequency ω describes its periodicity, similarly as for circular motion. The oscillation can be considered as projection of constant circular motion to a straight line. So we can analogically define the frequency f and period T :

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Simple harmonic motion VII

- Since the formula for the displacement is a result of two integrations is contains two integration parameters :
  - the amplitude x_0, which is the maximal possible displacement and
  - the phase φ, which allows for describing oscillations with any displacement at zero time in the beginning
  - the integration parameters can be obtained from the boundary conditions.

Simple harmonic motion VIII

- Let's examine the time dependence of displacement, speed and acceleration :

$$x(t) = x_0 \sin(\omega t + \varphi) = x_0 \sin(\sqrt{\tfrac{k}{m}}t + \varphi)$$

$$v(t) = \dot{x}(t) = x_0 \omega \cos(\omega t + \varphi) = x_0 \sqrt{\tfrac{k}{m}} \cos(\sqrt{\tfrac{k}{m}}t + \varphi)$$

$$a(t) = \ddot{x}(t) = -x_0 \omega^2 \sin(\omega t + \varphi) = -x_0 \tfrac{k}{m} \sin(\sqrt{\tfrac{k}{m}}t + \varphi)$$

Simple harmonic motion IX

- All these quantities are harmonic with the same angular frequency but different amplitude and phase :
  - The speed advances the displacement by the quarter of the period. It has a maximum in the equilibrium point for zero displacement.
  - The acceleration goes against the displacement which exactly corresponds to the behaviour of the restoring force.

Simple harmonic motion X

- Let's examine the time-dependence of energy. It has apparently two components :
  - Kinetic E_k, since the oscillating particle moves with some time-dependent speed and
  - Potential E_p, since work has to be done to reach certain displacement and it is conserved since we neglect the energy losses in this case.

Simple harmonic motion XI

$$E_p(x) = \frac{kx^2}{2} = \frac{k}{2} x_0^2 \sin^2(\omega t + \varphi)$$

$$E_k(x) = \frac{m\dot{x}^2}{2} = \frac{m}{2} \omega^2 x_0^2 \cos^2(\omega t + \varphi)$$

- Let's substitute for the ω in the E_k :

$$E_k(x) = \frac{k}{2} x_0^2 \cos^2(\omega t + \varphi)$$

Simple harmonic motion XII

- Then for the total energy we get :

$$E(t) = E_k(t) + E_p(t) =$$

$$\frac{k}{2} x_0^2 [\cos^2(\omega t + \varphi) + \sin^2(\omega t + \varphi)]$$

- Using the identity $\sin^2 \alpha + \cos^2 \alpha = 1$ we get :

$$E(t) = \frac{k}{2} x_0^2 = \frac{m\omega^2 x_o^2}{2} = konst.$$

Simple harmonic motion XIII

- Let's sum the important properties we have found :
  - the kinetic energy is in-phase with the absolute value of the speed. So it doesn't depend on its direction.
  - the potential energy is in-phase with the absolute value of the displacement. Again, doesn't depend on its direction.
  - the total energy doesn't depend on time - during one oscillation the kinetic energy is gradually changing into the potential energy and back.

Simple harmonic motion XIV

- The total energy of oscillating system is given by the boundary conditions and it is conserved.
  - For instance: We pull the particle to a certain starting displacement which will then be the amplitude. We have to do some work to accomplish this and thereby we give the oscillator the starting total energy.
  - Or we can kick the particle, which was at rest, to give it some momentum and kinetic energy which will also be the starting total energy of the oscillator.
  - We can combine these methods and the starting total energy can have both kinetic and potential component.

Examples: Physical pendulum I

- Physical pendulum is any extended body that swings back and forth usually in gravitational field.
- The exception is a torsion pendulum that doesn't need the force of gravity.
- Let's have a rigid body that can rotate (at least to some extent) around horizontal axis which is at nonzero distance a from its center of mass.
- The equilibrium position is when the mass centre is bellow the axis.

Physical pendulum II

- Angular frequency of the pendulum ω is :

$$\omega = \sqrt{\frac{Ga}{J}}$$

here G is the weight and J the moment of inertia

- In the numerator we have 'elastic' properties which boost the motion and in the denominator are 'inertial' properties that try to slow it down.
- Penduli are used to measure time or the force of gravity.

A simple (mathematical) pendulum

- A special case of physical pendulum is the (simple) mathematical pendulum which has all its mass *m* concentrated in distance *l* from the axis of rotation ona lightweight cord.
- We can use formula for physical pendulum into which we insert : $a = l,\ G = mg,\ J = m\, l^2$.
- For the angular frequency ω and period T we get :

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{Ga}{J}} = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}; \quad T = 2\pi\sqrt{\frac{l}{g}}$$

Physical pendulum III

- A reduced length λ is being used for the physical pendulum. It is equal to the length of mathematical pendulum with the same period :

$$T = 2\pi \sqrt{\frac{J}{Ga}} = \sqrt{\frac{\mu}{g}} \Rightarrow \mu = \frac{J}{ma}$$

Interference of oscillations I

- If more restoring forces act the particle can move in several oscillations at the same time.
- In general the total oscillation is the superposition of the individual oscillations and the displacement is the superposition of individual displacements.
- Often we are interested under what conditions would the resulting oscillation be periodic or even harmonic.
- Let's now consider just two interfering oscillations.

Interference of oscillations II

- It is interesting that even if the interfering oscillations are harmonic the result motion is generally aperiodic. Especially:
  - If one frequency is a rational factor of the other the result oscillation is periodic.
  - Only if the frequencies are equal the result oscillation is harmonic.
- Let's look at several special and important cases of interference of harmonic oscillations.

Superposition of linear oscillations I

- If both interfering oscillations have the same frequency they can differ only in the amplitude and phase. Their superposition:
  - has again the same frequency as each oscilation
  - its amplitude and phase can be calculated using two dimensional vectors or complex numbers.

Superposition of linear oscillations II

- We can prove this statement for two oscillations which we can then generalize for more oscillations.
- Let our oscillations be described by tha parameters x_{10}, φ_1 a x_{20}, φ_2. then :

$$x(t) = x_{10}\cos(\omega t + \varphi_1) + x_{20}\cos(\omega t + \varphi_2)$$

- cosines can be cosiny decomposed by well know formulas, reorganized and assembled back again :

Superposition of linear oscillations III

- $$x(t) = x_{10}\cos\omega t\cos\varphi_1 - x_{10}\sin\,\omega t\sin\,\varphi_1 +$$
  $$x_{20}\cos\omega t\cos\varphi_2 - x_{20}\sin\,\omega t\sin\,\varphi_2 =$$
  $$\cos\omega t(x_{10}\cos\varphi_1 + x_{20}\cos\varphi_2) -$$
  $$\sin\,\omega t(x_{10}\sin\,\varphi_1 + x_{20}\sin\,\varphi_2) =$$
  $$\cos\omega t(x_{120}\cos\varphi) - \sin\,\omega t(x_{120}\sin\,\varphi) =$$
  $$x_{120}\cos(\omega t + \varphi)$$

Superposition of linear oscillations IV

- The result oscillation have the angle frequency ω, the same as the original oscillations, the same as superposition, amplitude x_{120} and phase φ.
- The amplitude and phase are the results of these of the original oscillations.
- Since each oscillation is described by two parameters we use two-dimensional mathematical apparatus, either using special 2D vectors – the phasors or complex numbers.
- We illustrate the phasor approach :

Superposition of linear oscillations V

- From the previous we see that the first component of the result oscillation is the sum of the first components of the original oscillations :

$$x_{120} \cos \varphi = x_{10} \cos \varphi_1 + x_{20} \cos \varphi_2$$

and similarly the second one :

$$x_{120} \sin \varphi = x_{10} \sin \varphi_1 + x_{20} \sin \varphi_2$$

- The corresponds exactly to the sum of vectors.

Superposition of linear oscillations VI

- Quite an interesting situation is when the oscillations have near angular frequencies. For simplicity we can assume the same amplitude and phase :

$$x(t) = x_0 \cos \omega_1 t + x_0 \cos \omega_2 t =$$

$$2x_0 \cos \frac{\omega_1 + \omega_2}{2} t \cos \frac{\omega_1 - \omega_2}{2} t$$

Superposition of linear oscillations VII

- The result oscillation :
  - has angle frequency equal to the average frequency of the original oscillations, all these frequencies are comparable
  - has the amplitude modulated by the difference of the original frequencies, which is typically low. This appearing and fading of tone is called beats in acoustic.

Waves I

- Imagine a long straight row of bound oscillators – mass points with a special property that each of them can perform oscillations around its equilibrium position. The bonds between them can be described using e.g. by Young modules E a G. If one of these oscillators is pulled from its equilibrium position and released it starts to oscillate and these oscillations spread through the row successively in both directions. This motion of displacement in both space and time is called a wave.
- According the character of the bonds the wave motion can be :
  - transversal – with the displacement perpendicular to the direction of the wave motion – needs nonzero shear modulus
  - longitudinal – with the displacement in the direction of the wave motion
  - a superposition of both

Waves II

- Since each oscillator oscillates around its equilibrium mass doesn't spread trough the space while energy does – typical property of waves
- For simplicity we deal with a special harmonic wave where the displacement spreads by the speed *c* from the origin along the *x-axis* and it is a harmonic function of both the coordinate *x* and the time *t* :

$$\vec{u}(x,t) = \vec{u}_0 \cos\omega\left(t \mp \frac{x}{c}\right)$$

  - the displacement has both the transversal and longitudinal component but from now we consider just the magnitude $u(x,t)$
  - the sign "-" holds for the positive part of the *x-axis*
  - the displacement in the point *x* is the same as was in the origin before the time $\tau = x/c$ it took the wave to reach the point *x*

Waves III

- The displacement of any wave is a solution of the general Laplace equation :

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

  - so do also our harmonic waves spreading in the space of harmonic oscillators

Waves IV

- We can easily illustrate the periodicity in both time and space :

$$u(x,t) = u_0 \cos \frac{2\pi}{T}(t \mp \frac{x}{c}) = u_0 \cos 2\pi(\frac{t}{T} \mp \frac{x}{\lambda})$$

here $\lambda = cT$ is the wavelength the distance to which the wave gets during one period T. So T describes the periodicity in time and λ in space.

Waves V

- Using the definition of λ the following holds :

$$cT = \lambda \Rightarrow \frac{c}{f} = \lambda \Rightarrow c = f\lambda$$

- In spectroscopy the inverse wavelength that is number of waves per the unit of length is being used :

$$\sigma = \frac{1}{\lambda}$$

it is apparently the space analog of the frequency.

Waves VI

- The space-analog of the angular frequency is the wave number : $$k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

  employing k we get very compact equation :

$$u(x,t) = u_0 \cos(\omega t \mp \frac{2\pi x}{Tc}) = u_0 \cos(\omega t \mp kx)$$

- We can further write:

$$\omega = 2\pi f = 2\pi\sigma c = \frac{2\pi}{\lambda} c = kc$$

Waves VII

- In the 3D space we can use the wave vector

$$\vec{k} = \frac{2\pi}{\lambda}\vec{k}_0$$

to describe fully a plane wave including the direction of its spread where the unit vector $\vec{k}_0$ points. Then for the displacement at some point $\vec{r}$ we write :

$$\vec{u}(\vec{r},t) = \vec{u}_0 \cos(\omega t - \vec{k}\cdot\vec{r})$$

The Speed of Waves I

- Let's have a wave running e.g. in a string.
  - The maximum of amplitude is a 'hill' the top of which is moving with a speed c *c*.
  - An element of length at the top moves on a path that can be approximated by a piece of a circle.
  - The forces on this piece must add to make the centripetal force to allow this movement.

The Speed of Waves II

$$2F\sin\varphi = \frac{\Delta m c^2}{R}$$

- We assume a small angle which we substitute by parameters of the circle and the mass of the element we express using the linear density (density per the unit of length) μ :

$$2F\sin\varphi \cong 2F\varphi = \frac{F\Delta l}{R} = \frac{\Delta l \mu c^2}{R}$$

So :

$$c = \sqrt{\frac{F}{\mu}}$$

The Speed of Waves III

- Similarly as in the formulas for the angle frequency, in the formula the speed of waves :
  - quantities related to elastic properties of space where waves move matter are in the numerator
  - in the denominator are the inertia properties.
- The formula for the speed in continuum is: $c = \sqrt{\dfrac{K}{\rho}}$
- Here K is the module of volume elasticity and γ the coefficient of compressibility defined earlier :

$$K = \frac{1}{\gamma} = \frac{-dp}{\dfrac{dV}{V}}$$

The Energy of Waves I

- The wave motion are successive oscillations of single oscillators along the direction of spreading.
  - It can be expected that both the kinetic and potential energies will spread.
  - As in the case of oscillations the mean kinetic and mean potential energies are the same and thereby each is half of the total energy.
- The mean kinetic energy per the unit of length is :

$$< E_k > = \tfrac{1}{4}\,\mu\omega^2 {u_0}^2$$

The Energy of Waves II

- This energy is carried together with the potential energy through an element of length with speed c so by time derivative we obtain the carried power :

$$< P > = \frac{d[(< E_k > + < E_p >)ct]}{dt} = \tfrac{1}{2}\mu c \omega^2 u_0{}^2$$

- The power depends on
  - parameters of the medium μ and c
  - properties of the wave ω^2 and $u_0{}^2$

Composition of Waves I

- When composing waves the principle of superposition holds as well as in the case for oscillations.
- Expressed mathematically we use vector sum or in special cases normal sum.
- For two waves both transversal or both longitudinal spreading along the x-axis :

$$u(x,t) = u_1(x,t) + u_2(x,t)$$

Composition of Waves II

- For the character of the resulting waves similar rules hold as for oscillations, but :
  - waves have generally four parameters e.g. the amplitude, the angular frequency, the wave number and the phase. If we further assume the same phase :

$$u(x,t) = u_{10}\cos(\omega_1 t - k_1 x) + u_{20}\cos(\omega_2 t - k_2 x)$$

Composition of Waves III

- If in a certain medium the speed of waves doesn't depend on frequency the number of independent parameters reduces to two. Then the rules for composition of waves are exactly the same as for oscillations.
- In real media the speed of waves generally depends on frequency. This effect is called dispersion. In optics, for instance, it is the reason for color aberration of lenses.

Composition of Waves IV

- If harmonic waves have the same speed but different amplitude and angle frequency the resulting composed wave is generally aperiodic.
- But depending on the relation of their angle frequencies it can also be periodic and even harmonic.

Composition of Waves V

- An important and frequent case occurs when the waves have the same angle frequency. Then :
  - In each point oscillations of the same frequency add.
  - The resulting oscillation has the same frequency and a certain amplitude and phase.
  - For simplicity we further assume the same unit amplitude for both waves.

Composition of Waves VI

$$u(x,t) = \cos(\omega t - kx) + \cos(\omega t - kx + \varphi) =$$

$$2\cos\tfrac{\varphi}{2}\cos(\omega t - kx + \tfrac{\varphi}{2})$$

- The result wave is modulated by a term that depends on the mutual phase shift.
- The extremes are the most important
  - constructive – both waves are exactly in-phase and their amplitudes add so if they are the same the result wave is twice the original amplitude
  - destructive – both waves are exactly out-of-phase and their amplitudes subtract so if they are the same their amplitudes cancel

Standing waves I

- A special case appears when the wave is composed with its own reflection on some obstacle

$$\cos(\omega t - kx) + \cos(\omega t + kx + \varphi) =$$

$$2\cos(kx + \tfrac{\varphi}{2})\cos(\omega t + \tfrac{\varphi}{2})$$

  - The phase shift describes that
    - depending on the obstacle the phase can change
    - the obstacle can be in various distances

Standing waves II

- At suitable conditions the result wave is stable in time – a standing wave. Then it is a function of the space (coordinate) only. It has :
  - antinodes – places with maximum amplitude
  - nodes– places with zero amplitude
- Function of all mechanical musical instruments is based on standing waves due to constructive interference of a wave and its reflection.

Standing waves III

- The conditions for the existence of standing waves are fulfilled by the natural or resonant frequencies the lowest of them is the fundamental frequency the higher are its integer multiples – higher harmonics.
- The relative intensities of harmonics make the color of the sound of the instrument. The differences exist exit for many reasons. For instance they are nodes at both ends of a string but one node and one antinode in the case of a flute.

Composition of Waves VII

- More waves with specially related angular frequencies can add to form a wave of a special shape. This is called the Fourier analysis :

$$u(x,t) = \sum_n a_n \cos(n\omega t - kx)$$

- So e.g. a saw-saw tooth pulse is obtained by :

$$u(t) = \sum_n -\tfrac{1}{n} \sin(n\omega t)$$

Doppler Effect I

- When we hear an ambulance moving through nearby streets we know that the frequency of its siren shifts and from our experience we recognize when it approaches us, when it leaves or even whether it has stopped.
- Let's study this in detail. We describe the motion of :
  - the source of (sound) waves by the velocity v
  - the receiver of waves by the velocity u
  - the medium which waves spread by the velocity w
  - Assume that the speed of waves c is more than u, v, w, but much less than the speed of light in vacuum.
  - The velocities are positive when in the direction of $+x$ axis.

Doppler Effect II

- At first suppose that the source as well as the medium are at rest ($v = w = 0$) near the origin and the receiver is at the right of the origin moving away by a velocity $u > 0$.
- The frequency which the receiver hears (the pitch) depends on the number of waves that pass him per second. See the conveyor belt Modern Times by Charles Chaplin!
  - if he was also at rest : $f_0 = \dfrac{c}{\lambda_0}$

Doppler Effect III

- if the receiver moves, waves pass him not with the velocity c but the relative velocity c - u. So using the previous :

$$f_u = \frac{c-u}{\lambda_0} = \frac{c-u}{c} f_0 \Rightarrow \frac{f_u}{f_0} = \frac{c-u}{c}$$

- if the receiver moves away ($u > 0$) the frequency is lower, if he approaches if is higher.

Doppler Effect IV

- let now the medium and the receiver be at rest. The receiver is at the right but at some distance from the origin. And the source is close to the origin and moves right $v>0$ so it approaches the receiver.
- during one period T_0 the source sends one wavelength
- at the moment he is sending the end of the wave he is at the distance $T_0 v$ from the point where he started to send its beginning. The beginning got into the distance $T_0 c$ so the new wavelength is squeezed into $T_0(c-v)$. So:

$$\lambda_v = T_0(c-v) \Rightarrow \frac{f_v}{f_0} = \frac{\lambda_0}{\lambda_v} = \frac{T_0 c}{T_0(c-v)}$$

- for the source moving away from the receiver $v<0$ the frequency is lower, if it approached him $v>0$ the frequency would be higher.

Doppler Effect V

- if only the medium moves uniformly $w<>0$, its velocity simply adds to the speed of the waves and if u=v=0 the frequency doesn't change :

$$\lambda_w = T_0(c+w) \Rightarrow f_w = \frac{c+w}{\lambda_w} = f_0$$

- if however also the receiver moves $w<>0$, $u<>0$, $v=0$, then the motion of medium makes change :

$$f_w = \frac{c+w-u}{\lambda_w} = f_0 \frac{c+w-u}{c+w} \Rightarrow \frac{f_w}{f_0} = \frac{c+w-u}{c+w}$$

Doppler Effect VI

- Similarly if the medium and the source move $w<>0,\ u=0,\ v<>0$:

$$\lambda_{vw} = T_0(c+w-v) \Rightarrow \frac{f_v}{f_0} = \frac{\lambda_0}{\lambda_v} = \frac{c+w}{c+w-v}$$

- Now we are ready to write general formula valid for any situation :

$$\frac{f_{uvw}}{f_0} = \frac{c+w-u}{c+w-v}$$

Doppler Effect VII

- A possible disadvantage of our notation is that from the sigh of u it is not clear whether the receiver moves away or approaches.
- It is necessary to look for the relative velocity $u - v$ but that is in accord with the real situation.
- Our notation is however consistent with the standard use of sighs for velocities and mainly the formulas are unambiguous.
- Note that if the source and the a receiver approach each other the frequency increases but it matters who moves and who is at rest. This asymmetry is not due to the notation but it is real: If the source moves the waves are deformed in space while if it is at rest they are not.

Doppler Effect VIII

- Finally let's assume medium at rest and the speeds of the source or the receiver negligible compared to the speed of waves. Then :

$$\frac{f_{uv}}{f_0} = \frac{c-u}{c-v} = \frac{c-v+(v-u)}{c-v} = 1 + \frac{v-u}{c-v} \cong 1 + \frac{v-u}{c}$$

  - this relation is already symmetric.
  - $v - u$ is the relative speed, positive for approaching
  - this works also for electromagnetic waves (light)

The Speed of Sound in a Spring I

Let's have a string with a length $l = 1\ m$, mass $m = 4\ g$ stretched by a force $F = 10\ N$.

What is the speed of sound in the string and at which frequencies will the string play?

The linear density of the string is je $\mu = 4\ 10^{-3}$ kg/m so the speed of the sound is :

$$c = \sqrt{\frac{F}{\mu}} = \sqrt{2500} = 50\, ms^{-1}$$

The frequencies can be found from the formula :

$$c = f\lambda$$

The Speed of Sound in a Spring II

The wavelengths will be those at which there is a constructive interference of waves running in both direction when standing waves form.

Waves in string must have nodes on both ends.

Different situation would be e.g. in the case of a flute which is opened at one end. There is a node at one and an antinode at the other end.

The Speed of Sound in a Spring III

The fundamental frequency corresponds to the longest such wave. The other tones are integer multiplies of this frequency - the (higher) harmonics :

$$\lambda_1 = 2l; \;\; \lambda_2 = l, \;\; \lambda_3 = \frac{2l}{3} \Rightarrow$$

$$\lambda_n = \frac{2l}{n} \Rightarrow f_n = \frac{c}{2l} n$$

For our string :

$$f_n = \frac{c}{2l} n = 25\, n\, Hz$$

The Speed of Sound in Water I

By means of small explosions the speed of waves in the sea water was found to be $c = 1.43\ 10^3\ m/s$.

What is the compressed of water in the greatest depth on the Earth?

From the density of the sea water $\rho = 1.03\ 10^3\ kg\ m^{-3}$ and the speed of sound the modulus of compressibility K is :

$$c = \sqrt{\frac{K}{\rho}} \Rightarrow K = \rho c^2 = 2.1 GPa$$

The Speed of Sound in Water II

The compressibility factor then is :

$$\gamma = \frac{\Delta V}{V}\frac{1}{p} = \frac{1}{K} = 4.7510^{-10}\ Pa^{-1}$$

and relative compression: $$\frac{\Delta V}{V} = \gamma\, p \approx 5.10^{-10}\ p$$

The relative compression at the ambient pressure *10^5 Pa* tthen is *$5\ 10^{-5}$* . At the bottom of the Mariana trench at the pressure *~10^8 Pa*, it is roughly *5%*. Water is not an ideal liquid. If it was the sound would spread in it with infinite speed!

The physical pendulum I

Let's have physical pendulum the mass centre of which is in the distance a under the horizontal axis of rotation.

If we displace the pendulum by a small angle φ, a restoring torque appears.

The equation of motion is:

$$T(\varphi) = -Ga\sin\varphi = J\varepsilon = J\ddot{\varphi}$$

The physical pendulum II

For small swings we can suppose sin(φ) ≈ φ [rad] and solution of a simplified equation:

$$J\ddot{\varphi} = -Ga\varphi$$

are harmonic oscillations:

$$\varphi(t) = \varphi_0 \sin(\omega t + \psi) = \varphi_0 \sin(\sqrt{\tfrac{Ga}{J}}t + \psi)$$

with the angular frequency :

$$\omega = \sqrt{\frac{Ga}{J}}$$

A simple pendulum

After substituting $a = l$, $G = mg$ a $J = ml^2$ in the formula for the physical pendulum we get:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{Ga}{J}} = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$$

And for the period : $$T = 2\pi\sqrt{\frac{l}{g}}$$

The Time and Space Periodicity

From the periodicity of the function *cos* we can easily show that in the time which is an integer multiple of the period mT before or after some time t_1 the displacement is the same. So the displacement in the times t_1 and $t_2 = t_1 + mT$ is the same:

$$\cos 2\pi\left(\frac{t+mT}{T} \mp \frac{x}{\lambda}\right) = \cos 2\pi\left(m + \frac{t}{T} \mp \frac{x}{\lambda}\right) = \cos 2\pi\left(\frac{t}{T} \mp \frac{x}{\lambda}\right)$$

The same is true for coordinates x_1 and $x_2 = x_1 + n\lambda$, where n it an integer :

$$\cos 2\pi\left(\frac{t}{T} \mp \frac{x+n\lambda}{\lambda}\right) = \cos 2\pi\left(\frac{t}{T} \mp \frac{x}{\lambda} \pm n\right) = \cos 2\pi\left(\frac{t}{T} \mp \frac{x}{\lambda}\right)$$

The Mean Kinetic Energy of Waves I

The kinetic energy of an element of the length dx of the wave depends on the speed of oscillations. If u is the displacement then :

$$dE_k = \frac{dm\dot{u}^2}{2} = \tfrac{1}{2}\mu dx(-\omega u_0)^2 \sin^2(\omega t - kx)$$

Then the mean kinetic energy per the unit of length is obtained by the integration :

$$< E_k >= \tfrac{1}{\lambda}\int_0^{\lambda} dE_k = \tfrac{1}{4}\mu\omega^2 u_0{}^2$$

^

The Mean Kinetic Energy of Waves II

By decomposition using the formula

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2\sin^2 \alpha$$

we get two more simple integrals and the second of them is zero :

$$\tfrac{1}{\lambda}\int_0^\lambda dE_k = \frac{\mu\omega^2 u_0^2}{2\lambda}\int_0^\lambda \sin^2(\omega t - kx)dx =$$

$$\frac{\mu\omega^2 u_0^2}{4\lambda}[\int_0^\lambda dx - \int_0^\lambda \cos 2(\omega t - kx)]dx =$$

$$\frac{\mu\omega^2 u_0^2}{4\lambda}[x]_0^\lambda = \frac{\mu\omega^2 u_0^2}{4}$$

The Mean Kinetic Energy of Waves III

We can prove that the second integral is really zero :

$$-\int_0^\lambda \cos 2(\omega t - kx)]dx = \tfrac{1}{2k}[\sin\, 2(\omega t - kx)]_0^\lambda =$$

$$\tfrac{1}{2k}[\sin\, 2(\omega t - k\lambda) - \sin\, 2(\omega t - 0)] =$$

$$\tfrac{1}{2k}[\sin\, 2(\omega t - 2\pi) - \sin\, 2(\omega t - 0)] = 0$$

In the one but last step we have substituted for the wave number $k=2\pi/\lambda$ and then used the periodicity of the function *sin*.

Physics

8

Oscillations and Waves

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Mains Topics

- Oscillations
  - general, periodic, harmonic, damped, un-damped
- Harmonic oscillations
  - equation of motion and its solutions
  - time dependence of displacement, speed, acceleration, potential and kinetic energy
  - adding oscillations, damped and forced oscillations
- Waves – general and harmonic waves
  - description, periodicity in space and time, transport of energy
  - standing waves, interference, Doppler's effect

Introduction – oscillations and waves I

- Studies of oscillatory or vibration motion have to be added to the previously studied translation and circular movements.
  - their existence stems from the elastic character of interactions between particles of matter
  - they are widely spread in Nature
  - Oscillation energy is an important type of energy

Intro – oscillations and waves II

- Generally oscillation of particle is movement that is limited in space.

  Wave is spreading of vibration movement in space that carries energy but not mass
- Should the mass particle oscillate the following items must exist :
  - equilibrium position where forces don't act on the particle
  - restoring forces – try to return the particle to the e.p.
  - the equilibrium must be stable

Type of oscillations I

- Character of oscillations, the precise time dependence of the displacement is given by :
  - the character of restoring forces - their actual dependence on the displacement. The force field is often conservative and the forces follow from the potential.
  - possible existence of dissipative forces.
- An important subset are periodic oscillations, where the time dependence of displacement repeats periodically.
- An important subset of periodic osc. are harmonic oscillations where the time dependence of displacement can be expressed using a harmonic = goniometric function of time.

Type of oscillations II

- Undamped periodic and also harmonic oscillations can exist only if there are no losses of mechanical energy present.
- If dissipative forces are present the mechanical energy decreases and eventually the movement stops. This is typical for damped oscillations.

Type of oscillations III

- Strictly speaking all real oscillations should be damped.
- Undamped oscillations still are important since:
    - damping can be small or negligible
    - losses of energy can be restored convenient way

Type of oscillations IV

- Harmonic oscillations are very important since:
  - the special restoring forces they need do exist.
  - any periodic oscillation can be expressed as Fourier series of harmonic oscillations
  - any general oscillation can be expressed as Fourier integral of harmonic oscillations
  - Since both addition and integration are linear operations many properties of harmonic oscillations are valid more generally

Simple (undamped) harmonic motion I

- Let's consider the following simple situation :
  - a particle can move only on a straight line which we coincide with the x-axis
  - the equilibrium position is chosen as the origin
  - the restoring force is directly proportional to the displacement

$$F(x) = -kx$$

Simple harmonic motion II

- At first we find work necessary to reach the displacement x moving from the origin :
  - since the restoring force depends on the current displacement χ we have to integrate :

$$E_p(x) = W(x) = \int_0^x k\chi d\chi = \frac{kx^2}{2}$$

  - this result is symmetric: To reach some displacement x either negative of positive we have to supply positive work
  - and it corresponds to parabolic potential well

Simple harmonic motion III

- This type of restoring force can exist in any field e.g. in gravitational or electrostatic which have potential and the potential well can be approximated by parabola at least for small oscillations.
- This restoring force may correspond to Hook's behavior. In this case the proportionality parameter k is proportional to the appropriate Young modulus and it is called the spring constant.

Simple harmonic motion IV

- If we substitute for force from the Newton's 2nd law we get the equation of motion, particularly differential equation of the second order with missing first order :

$$F = m\ddot{x} = -kx$$

- We guess the form of the solution and find its parameters :

$$x(t) = x_0 \sin(\omega t + \varphi)$$

Simple harmonic motion V

- Let's calculate its (first and) second time derivative and substitute it to the original equation : $\dot{x}(t) = x_0\omega\cos(\omega t + \varphi)$

$$\ddot{x}(t) = -\omega^2 x_0 \sin(\omega t + \varphi) = -\omega^2 x(t)$$

$$-m\omega^2 x(t) = -kx(t)$$

- From which we get : $\omega = \sqrt{\dfrac{k}{m}}$

Simple harmonic motion VI

- the time dependence of the displacement is :

$$x(t) = x_0 \sin(\omega t + \varphi) = x_0 \sin\left(\sqrt{\frac{k}{m}}t + \varphi\right)$$

  - angular frequency ω describes its periodicity, similarly as for circular motion. The oscillation can be considered as projection of constant circular motion to a straight line. So we can analogically define the frequency f and period T :

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Simple harmonic motion VII

- Since the formula for the displacement is a result of two integrations is contains two integration parameters :
  - the amplitude x_0, which is the maximal possible displacement and
  - the phase φ, which allows for describing oscillations with any displacement at zero time in the beginning
  - the integration parameters can be obtained from the boundary conditions.

Simple harmonic motion VIII

- Let's examine the time dependence of displacement, speed and acceleration :

$$x(t) = x_0 \sin(\omega t + \varphi) = x_0 \sin(\sqrt{\tfrac{k}{m}}t + \varphi)$$

$$v(t) = \dot{x}(t) = x_0 \omega \cos(\omega t + \varphi) = x_0 \sqrt{\tfrac{k}{m}} \cos(\sqrt{\tfrac{k}{m}}t + \varphi)$$

$$a(t) = \ddot{x}(t) = -x_0 \omega^2 \sin(\omega t + \varphi) = -x_0 \tfrac{k}{m} \sin(\sqrt{\tfrac{k}{m}}t + \varphi)$$

Simple harmonic motion IX

- All these quantities are harmonic with the same angular frequency but different amplitude and phase :
  - The speed advances the displacement by the quarter of the period. It has a maximum in the equilibrium point for zero displacement.
  - The acceleration goes against the displacement which exactly corresponds to the behaviour of the restoring force.

Simple harmonic motion X

- Let's examine the time-dependence of energy. It has apparently two components :
  - Kinetic E_k, since the oscillating particle moves with some time-dependent speed and
  - Potential E_p, since work has to be done to reach certain displacement and it is conserved since we neglect the energy losses in this case.

Simple harmonic motion XI

$$E_p(x) = \frac{kx^2}{2} = \frac{k}{2} x_0^2 \sin^2(\omega t + \varphi)$$

$$E_k(x) = \frac{m\dot{x}^2}{2} = \frac{m}{2} \omega^2 x_0^2 \cos^2(\omega t + \varphi)$$

- Let's substitute for the ω in the E_k :

$$E_k(x) = \frac{k}{2} x_0^2 \cos^2(\omega t + \varphi)$$

Simple harmonic motion XII

- Then for the total energy we get :

$$E(t) = E_k(t) + E_p(t) =$$

$$\frac{k}{2} x_0^2 [\cos^2(\omega t + \varphi) + \sin^2(\omega t + \varphi)]$$

- Using the identity $\sin^2\alpha + \cos^2\alpha = 1$ we get :

$$E(t) = \frac{k}{2} x_0^2 = \frac{m\omega^2 x_o^2}{2} = konst.$$

Simple harmonic motion XIII

- Let's sum the important properties we have found :
  - the kinetic energy is in-phase with the absolute value of the speed. So it doesn't depend on its direction.
  - the potential energy is in-phase with the absolute value of the displacement. Again, doesn't depend on its direction.
  - the total energy doesn't depend on time - during one oscillation the kinetic energy is gradually changing into the potential energy and back.

Simple harmonic motion XIV

- The total energy of oscillating system is given by the boundary conditions and it is conserved.
  - For instance: We pull the particle to a certain starting displacement which will then be the amplitude. We have to do some work to accomplish this and thereby we give the oscillator the starting total energy.
  - Or we can kick the particle, which was at rest, to give it some momentum and kinetic energy which will also be the starting total energy of the oscillator.
  - We can combine these methods and the starting total energy can have both kinetic and potential component.

Examples: Physical pendulum I

- Physical pendulum is any extended body that swings back and forth usually in gravitational field.
- The exception is a torsion pendulum that doesn't need the force of gravity.
- Let's have a rigid body that can rotate (at least to some extent) around horizontal axis which is at nonzero distance a from its center of mass.
- The equilibrium position is when the mass centre is bellow the axis.

Physical pendulum II

- Angular frequency of the pendulum ω is :

$$\omega = \sqrt{\frac{Ga}{J}}$$

  here G is the weight and J the moment of inertia
- In the numerator we have 'elastic' properties which boost the motion and in the denominator are 'inertial' properties that try to slow it down.
- Penduli are used to measure time or the force of gravity.

A simple (mathematical) pendulum

- A special case of physical pendulum is the (simple) mathematical pendulum which has all its mass m concentrated in distance l from the axis of rotation ona lightweight cord.
- We can use formula for physical pendulum into which we insert : $a = l,\ G = mg,\ J = m\, l^2$.
- For the angular frequency ω and period T we get :

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{Ga}{J}} = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}; \quad T = 2\pi\sqrt{\frac{l}{g}}$$

Physical pendulum III

- A reduced length λ is being used for the physical pendulum. It is equal to the length of mathematical pendulum with the same period :

$$T = 2\pi\sqrt{\frac{J}{Ga}} = \sqrt{\frac{\mu}{g}} \Rightarrow \mu = \frac{J}{ma}$$

Interference of oscillations I

- If more restoring forces act the particle can move in several oscillations at the same time.
- In general the total oscillation is the superposition of the individual oscillations and the displacement is the superposition of individual displacements.
- Often we are interested under what conditions would the resulting oscillation be periodic or even harmonic.
- Let's now consider just two interfering oscillations.

Interference of oscillations II

- It is interesting that even if the interfering oscillations are harmonic the result motion is generally aperiodic. Especially:
  - If one frequency is a rational factor of the other the result oscillation is periodic.
  - Only if the frequencies are equal the result oscillation is harmonic.
- Let's look at several special and important cases of interference of harmonic oscillations.

Superposition of linear oscillations I

- If both interfering oscillations have the same frequency they can differ only in the amplitude and phase. Their superposition:
  - has again the same frequency as each oscilation
  - its amplitude and phase can be calculated using two dimensional vectors or complex numbers.

Superposition of linear oscillations II

- We can prove this statement for two oscillations which we can then generalize for more oscillations.
- Let our oscillations be described by tha parameters x_{10}, φ_1 a x_{20}, φ_2. then :

$$x(t) = x_{10}\cos(\omega t + \varphi_1) + x_{20}\cos(\omega t + \varphi_2)$$

- cosines can be cosiny decomposed by well know formulas, reorganized and assembled back again :

Superposition of linear oscillations III

- $$x(t) = x_{10}\cos\omega t\cos\varphi_1 - x_{10}\sin\omega t\sin\varphi_1 +$$
$$x_{20}\cos\omega t\cos\varphi_2 - x_{20}\sin\omega t\sin\varphi_2 =$$
$$\cos\omega t(x_{10}\cos\varphi_1 + x_{20}\cos\varphi_2) -$$
$$\sin\omega t(x_{10}\sin\varphi_1 + x_{20}\sin\varphi_2) =$$
$$\cos\omega t(x_{120}\cos\varphi) - \sin\omega t(x_{120}\sin\varphi) =$$
$$x_{120}\cos(\omega t + \varphi)$$

Superposition of linear oscillations IV

- The result oscillation have the angle frequency ω, the same as the original oscillations, the same as superposition, amplitude x_{120} and phase φ.
- The amplitude and phase are the results of these of the original oscillations.
- Since each oscillation is described by two parameters we use two-dimensional mathematical apparatus, either using special 2D vectors – the phasors or complex numbers.
- We illustrate the phasor approach :

Superposition of linear oscillations V

- From the previous we see that the first component of the result oscillation is the sum of the first components of the original oscillations :

$$x_{120}\cos\varphi = x_{10}\cos\varphi_1 + x_{20}\cos\varphi_2$$

and similarly the second one :

$$x_{120}\sin\,\varphi = x_{10}\sin\,\varphi_1 + x_{20}\sin\,\varphi_2$$

- The corresponds exactly to the sum of vectors.

Superposition of linear oscillations VI

- Quite an interesting situation is when the oscillations have near angular frequencies. For simplicity we can assume the same amplitude and phase :

$$x(t) = x_0 \cos \omega_1 t + x_0 \cos \omega_2 t =$$

$$2x_0 \cos \frac{\omega_1 + \omega_2}{2} t \cos \frac{\omega_1 - \omega_2}{2} t$$

Superposition of linear oscillations VII

- The result oscillation :
  - has angle frequency equal to the average frequency of the original oscillations, all these frequencies are comparable
  - has the amplitude modulated by the difference of the original frequencies, which is typically low. This appearing and fading of tone is called beats in acoustic.

Waves I

- Imagine a long straight row of bound oscillators – mass points with a special property that each of them can perform oscillations around its equilibrium position. The bonds between them can be described using e.g. by Young modules E a G. If one of these oscillators is pulled from its equilibrium position and released it starts to oscillate and these oscillations spread through the row successively in both directions. This motion of displacement in both space and time is called a wave.
- According the character of the bonds the wave motion can be :
  - transversal – with the displacement perpendicular to the direction of the wave motion – needs nonzero shear modulus
  - longitudinal – with the displacement in the direction of the wave motion
  - a superposition of both

Waves II

- Since each oscillator oscillates around its equilibrium mass doesn't spread trough the space while energy does – typical property of waves
- For simplicity we deal with a special harmonic wave where the displacement spreads by the speed *c* from the origin along the *x-axis* and it is a harmonic function of both the coordinate *x* and the time *t* :

$$\vec{u}(x,t) = \vec{u}_0 \cos\omega(t \mp \tfrac{x}{c})$$

  - the displacement has both the transversal and longitudinal component but from now we consider just the magnitude $u(x,t)$
  - the sign "-" holds for the positive part of the *x-axis*
  - the displacement in the point x is the same as was in the origin before the time $\tau = x/c$ it took the wave to reach the point x

Waves III

- The displacement of any wave is a solution of the general Laplace equation :

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

  - so do also our harmonic waves spreading in the space of harmonic oscillators

Waves IV

- We can easily illustrate the periodicity in both time and space :

$$u(x,t) = u_0 \cos \frac{2\pi}{T}(t \mp \frac{x}{c}) = u_0 \cos 2\pi(\frac{t}{T} \mp \frac{x}{\lambda})$$

here $\lambda = cT$ is the wavelength the distance to which the wave gets during one period T. So T describes the periodicity in time and λ in space.

Waves V

- Using the definition of λ the following holds :

$$cT = \lambda \Rightarrow \frac{c}{f} = \lambda \Rightarrow c = f\lambda$$

- In spectroscopy the inverse wavelength that is number of waves per the unit of length is being used :

$$\sigma = \frac{1}{\lambda}$$

it is apparently the space analog of the frequency.

Waves VI

- The space-analog of the angular frequency is the wave number : $$k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

  employing k we get very compact equation :

$$u(x,t) = u_0 \cos(\omega t \mp \frac{2\pi x}{Tc}) = u_0 \cos(\omega t \mp kx)$$

- We can further write:

$$\omega = 2\pi f = 2\pi \sigma c = \frac{2\pi}{\lambda} c = kc$$

Waves VII

- In the 3D space we can use the wave vector

$$\vec{k} = \tfrac{2\pi}{\lambda}\vec{k}_0$$

to describe fully a plane wave including the direction of its spread where the unit vector $\vec{k}_0$ points. Then for the displacement at some point $\vec{r}$ we write :

$$\vec{u}(\vec{r},t) = \vec{u}_0 \cos(\omega t - \vec{k}\cdot\vec{r})$$

The Speed of Waves I

- Let's have a wave running e.g. in a string.
  - The maximum of amplitude is a 'hill' the top of which is moving with a speed c c.
  - An element of length at the top moves on a path that can be approximated by a piece of a circle.
  - The forces on this piece must add to make the centripetal force to allow this movement.

The Speed of Waves II

$$2F \sin \varphi = \frac{\Delta m c^2}{R}$$

- We assume a small angle which we substitute by parameters of the circle and the mass of the element we express using the linear density (density per the unit of length) μ :

$$2F \sin \varphi \cong 2F\varphi = \frac{F\Delta l}{R} = \frac{\Delta l \mu c^2}{R}$$

So :

$$c = \sqrt{\frac{F}{\mu}}$$

The Speed of Waves III

- Similarly as in the formulas for the angle frequency, in the formula the speed of waves :
  - quantities related to elastic properties of space where waves move matter are in the numerator
  - in the denominator are the inertia properties.
- The formula for the speed in continuum is: $c = \sqrt{\dfrac{K}{\rho}}$
- Here K is the module of volume elasticity and γ the coefficient of compressibility defined earlier :

$$K = \frac{1}{\gamma} = \frac{-dp}{\dfrac{dV}{V}}$$

The Energy of Waves I

- The wave motion are successive oscillations of single oscillators along the direction of spreading.
  - It can be expected that both the kinetic and potential energies will spread.
  - As in the case of oscillations the mean kinetic and mean potential energies are the same and thereby each is half of the total energy.
- The mean kinetic energy per the unit of length is :

$$\langle E_k \rangle = \tfrac{1}{4}\mu\omega^2 u_0{}^2$$

The Energy of Waves II

- This energy is carried together with the potential energy through an element of length with speed c so by time derivative we obtain the carried power :

$$< P >= \frac{d[(< E_k > + < E_p >)ct]}{dt} = \tfrac{1}{2}\,\mu c \omega^2 {u_0}^2$$

- The power depends on
  - parameters of the medium μ and c
  - properties of the wave ω^2 and $u_0{}^2$

Composition of Waves I

- When composing waves the principle of superposition holds as well as in the case for oscillations.
- Expressed mathematically we use vector sum or in special cases normal sum.
- For two waves both transversal or both longitudinal spreading along the x-axis :

$$u(x,t) = u_1(x,t) + u_2(x,t)$$

Composition of Waves II

- For the character of the resulting waves similar rules hold as for oscillations, but :
  - waves have generally four parameters e.g. the amplitude, the angular frequency, the wave number and the phase. If we further assume the same phase :

$$u(x,t) = u_{10}\cos(\omega_1 t - k_1 x) + u_{20}\cos(\omega_2 t - k_2 x)$$

Composition of Waves III

- If in a certain medium the speed of waves doesn't depend on frequency the number of independent parameters reduces to two. Then the rules for composition of waves are exactly the same as for oscillations.
- In real media the speed of waves generally depends on frequency. This effect is called dispersion. In optics, for instance, it is the reason for color aberration of lenses.

Composition of Waves IV

- If harmonic waves have the same speed but different amplitude and angle frequency the resulting composed wave is generally aperiodic.
- But depending on the relation of their angle frequencies it can also be periodic and even harmonic.

Composition of Waves V

- An important and frequent case occurs when the waves have the same angle frequency. Then :
  - In each point oscillations of the same frequency add.
  - The resulting oscillation has the same frequency and a certain amplitude and phase.
  - For simplicity we further assume the same unit amplitude for both waves.

Composition of Waves VI

$$u(x,t) = \cos(\omega t - kx) + \cos(\omega t - kx + \varphi) =$$

$$2\cos\tfrac{\varphi}{2}\cos(\omega t - kx + \tfrac{\varphi}{2})$$

- The result wave is modulated by a term that depends on the mutual phase shift.
- The extremes are the most important
  - constructive – both waves are exactly in-phase and their amplitudes add so if they are the same the result wave is twice the original amplitude
  - destructive – both waves are exactly out-of-phase and their amplitudes subtract so if they are the same their amplitudes cancel

Standing waves I

- A special case appears when the wave is composed with its own reflection on some obstacle

$$\cos(\omega t - kx) + \cos(\omega t + kx + \varphi) =$$

$$2\cos(kx + \tfrac{\varphi}{2})\cos(\omega t + \tfrac{\varphi}{2})$$

  - The phase shift describes that
    - depending on the obstacle the phase can change
    - the obstacle can be in various distances

Standing waves II

- At suitable conditions the result wave is stable in time – a standing wave. Then it is a function of the space (coordinate) only. It has :
  - antinodes – places with maximum amplitude
  - nodes– places with zero amplitude
- Function of all mechanical musical instruments is based on standing waves due to constructive interference of a wave and its reflection.

Standing waves III

- The conditions for the existence of standing waves are fulfilled by the natural or resonant frequencies the lowest of them is the fundamental frequency the higher are its integer multiples – higher harmonics.
- The relative intensities of harmonics make the color of the sound of the instrument. The differences exist exit for many reasons. For instance they are nodes at both ends of a string but one node and one antinode in the case of a flute.

Composition of Waves VII

- More waves with specially related angular frequencies can add to form a wave of a special shape. This is called the Fourier analysis :

$$u(x,t) = \sum_n a_n \cos(n\omega t - kx)$$

- So e.g. a saw-saw tooth pulse is obtained by :

$$u(t) = \sum_n -\tfrac{1}{n} \sin(n\omega t)$$

Doppler Effect I

- When we hear an ambulance moving through nearby streets we know that the frequency of its siren shifts and from our experience we recognize when it approaches us, when it leaves or even whether it has stopped.
- Let's study this in detail. We describe the motion of :
  - the source of (sound) waves by the velocity v
  - the receiver of waves by the velocity u
  - the medium which waves spread by the velocity w
  - Assume that the speed of waves c is more than u, v, w, but much less than the speed of light in vacuum.
  - The velocities are positive when in the direction of $+x$ axis.

Doppler Effect II

- At first suppose that the source as well as the medium are at rest ($v = w = 0$) near the origin and the receiver is at the right of the origin moving away by a velocity $u > 0$.
- The frequency which the receiver hears (the pitch) depends on the number of waves that pass him per second. See the conveyor belt Modern Times by Charles Chaplin!
  - if he was also at rest : $f_0 = \dfrac{c}{\lambda_0}$

Doppler Effect III

- if the receiver moves, waves pass him not with the velocity *c* but the relative velocity *c* - *u*. So using the previous :

$$f_u = \frac{c-u}{\lambda_0} = \frac{c-u}{c} f_0 \Rightarrow \frac{f_u}{f_0} = \frac{c-u}{c}$$

- if the receiver moves away (*u > 0*) the frequency is lower, if he approaches if is higher.

Doppler Effect IV

- let now the medium and the receiver be at rest. The receiver is at the right but at some distance from the origin. And the source is close to the origin and moves right $v>0$ so it approaches the receiver.
- during one period T_0 the source sends one wavelength
- at the moment he is sending the end of the wave he is at the distance $T_0 v$ from the point where he started to send its beginning. The beginning got into the distance $T_0 c$ so the new wavelength is squeezed into $T_0(c-v)$. So:

$$\lambda_v = T_0(c-v) \Rightarrow \frac{f_v}{f_0} = \frac{\lambda_0}{\lambda_v} = \frac{T_0 c}{T_0(c-v)}$$

- for the source moving away from the receiver $v<0$ the frequency is lower, if it approached him $v>0$ the frequency would be higher.

Doppler Effect V

- if only the medium moves uniformly $w<>0$, its velocity simply adds to the speed of the waves and if u=v=0 the frequency doesn't change :

$$\lambda_w = T_0(c+w) \Rightarrow f_w = \frac{c+w}{\lambda_w} = f_0$$

- if however also the receiver moves $w<>0$, $u<>0$, $v=0$, then the motion of medium makes change :

$$f_w = \frac{c+w-u}{\lambda_w} = f_0 \frac{c+w-u}{c+w} \Rightarrow \frac{f_w}{f_0} = \frac{c+w-u}{c+w}$$

Doppler Effect VI

- Similarly if the medium and the source move $w<>0,\ u=0,\ v<>0$:

$$\lambda_{vw} = T_0(c+w-v) \Rightarrow \frac{f_v}{f_0} = \frac{\lambda_0}{\lambda_v} = \frac{c+w}{c+w-v}$$

- Now we are ready to write general formula valid for any situation :

$$\frac{f_{uvw}}{f_0} = \frac{c+w-u}{c+w-v}$$

Doppler Effect VII

- A possible disadvantage of our notation is that from the sigh of u it is not clear whether the receiver moves away or approaches.
- It is necessary to look for the relative velocity $u - v$ but that is in accord with the real situation.
- Our notation is however consistent with the standard use of sighs for velocities and mainly the formulas are unambiguous.
- Note that if the source and the a receiver approach each other the frequency increases but it matters who moves and who is at rest. This asymmetry is not due to the notation but it is real: If the source moves the waves are deformed in space while if it is at rest they are not.

Doppler Effect VIII

- Finally let's assume medium at rest and the speeds of the source or the receiver negligible compared to the speed of waves. Then :

$$\frac{f_{uv}}{f_0} = \frac{c-u}{c-v} = \frac{c-v+(v-u)}{c-v} = 1 + \frac{v-u}{c-v} \simeq 1 + \frac{v-u}{c}$$

  - this relation is already symmetric.
  - $v-u$ is the relative speed, positive for approaching
  - this works also for electromagnetic waves (light)

The Speed of Sound in a Spring I

Let's have a string with a length $l = 1\ m$, mass $m = 4\ g$ stretched by a force $F = 10\ N$.

What is the speed of sound in the string and at which frequencies will the string play?

The linear density of the string is je $\mu = 4\ 10^{-3}$ kg/m so the speed of the sound is :

$$c = \sqrt{\frac{F}{\mu}} = \sqrt{2500} = 50\,ms^{-1}$$

The frequencies can be found from the formula :

$$c = f\lambda$$

The Speed of Sound in a Spring II

The wavelengths will be those at which there is a constructive interference of waves running in both direction when standing waves form.

Waves in string must have nodes on both ends.

Different situation would be e.g. in the case of a flute which is opened at one end. There is a node at one and an antinode at the other end.

The Speed of Sound in a Spring III

The fundamental frequency corresponds to the longest such wave. The other tones are integer multiplies of this frequency - the (higher) harmonics :

$$\lambda_1 = 2l; \ \lambda_2 = l, \ \lambda_3 = \frac{2l}{3} \Rightarrow$$

$$\lambda_n = \frac{2l}{n} \Rightarrow f_n = \frac{c}{2l} n$$

For our string :

$$f_n = \frac{c}{2l} n = 25 \, n \, Hz$$

The Speed of Sound in Water I

By means of small explosions the speed of waves in the sea water was found to be $c = 1.43\ 10^3\ m/s$.

What is the compressed of water in the greatest depth on the Earth?

From the density of the sea water $\rho = 1.03\ 10^3\ kg\ m^{-3}$ and the speed of sound the modulus of compressibility K is :

$$c = \sqrt{\frac{K}{\rho}} \Rightarrow K = \rho c^2 = 2.1 GPa$$

The Speed of Sound in Water II

The compressibility factor then is :

$$\gamma = \frac{\Delta V}{V}\frac{1}{p} = \frac{1}{K} = 4.7510^{-10}\ Pa^{-1}$$

and relative compression: $$\frac{\Delta V}{V} = \gamma\, p \approx 5.10^{-10}\ p$$

The relative compression at the ambient pressure 10^5 *Pa* tthen is $5\ 10^{-5}$. At the bottom of the Mariana trench at the pressure ~10^8 *Pa*, it is roughly *5%*. Water is not an ideal liquid. If it was the sound would spread in it with infinite speed!

The physical pendulum I

Let's have physical pendulum the mass centre of which is in the distance a under the horizontal axis of rotation.

If we displace the pendulum by a small angle φ, a restoring torque appears.

The equation of motion is:

$$T(\varphi) = -Ga\sin\varphi = J\varepsilon = J\ddot{\varphi}$$

The physical pendulum II

For small swings we can suppose sin(φ) ≈ φ [rad] and solution of a simplified equation:

$$J\ddot{\varphi} = -Ga\varphi$$

are harmonic oscillations:

$$\varphi(t) = \varphi_0 \sin(\omega t + \psi) = \varphi_0 \sin(\sqrt{\tfrac{Ga}{J}}\,t + \psi)$$

with the angular frequency :

$$\omega = \sqrt{\frac{Ga}{J}}$$

A simple pendulum

After substituting $a = l$, $G = mg$ a $J = ml^2$ in the formula for the physical pendulum we get:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{Ga}{J}} = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}$$

And for the period : $$T = 2\pi\sqrt{\frac{l}{g}}$$

The Time and Space Periodicity

From the periodicity of the function *cos* we can easily show that in the time which is an integer multiple of the period mT before or after some time t_1 the displacement is the same. So the displacement in the times t_1 and $t_2 = t_1 + mT$ is the same:

$$\cos 2\pi\left(\frac{t+mT}{T} \mp \frac{x}{\lambda}\right) = \cos 2\pi\left(m + \frac{t}{T} \mp \frac{x}{\lambda}\right) = \cos 2\pi\left(\frac{t}{T} \mp \frac{x}{\lambda}\right)$$

The same is true for coordinates x_1 and $x_2 = x_1 + n\lambda$, where n it an integer :

$$\cos 2\pi\left(\frac{t}{T} \mp \frac{x+n\lambda}{\lambda}\right) = \cos 2\pi\left(\frac{t}{T} \mp \frac{x}{\lambda} \pm n\right) = \cos 2\pi\left(\frac{t}{T} \mp \frac{x}{\lambda}\right)$$

The Mean Kinetic Energy of Waves I

The kinetic energy of an element of the length *dx* of the wave depends on the speed of oscillations. If *u* is the displacement then :

$$dE_k = \frac{dm\dot{u}^2}{2} = \tfrac{1}{2}\mu dx(-\omega u_0)^2 \sin^2(\omega t - kx)$$

Then the mean kinetic energy per the unit of length is obtained by the integration :

$$< E_k >= \tfrac{1}{\lambda}\int_0^{\lambda} dE_k = \tfrac{1}{4}\mu\omega^2 u_0^{\ 2}$$

The Mean Kinetic Energy of Waves II

By decomposition using the formula

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2\sin^2 \alpha$$

we get two more simple integrals and the second of them is zero :

$$\frac{1}{\lambda}\int_0^{\lambda} dE_k = \frac{\mu\omega^2 u_0^2}{2\lambda}\int_0^{\lambda} \sin^2(\omega t - kx)dx =$$

$$\frac{\mu\omega^2 u_0^2}{4\lambda}[\int_0^{\lambda} dx - \int_0^{\lambda} \cos 2(\omega t - kx)]dx =$$

$$\frac{\mu\omega^2 u_0^2}{4\lambda}[x]_0^{\lambda} = \frac{\mu\omega^2 u_0^2}{4}$$

The Mean Kinetic Energy of Waves III

We can prove that the second integral is really zero :

$$-\int_0^{\lambda} \cos 2(\omega t - kx)]dx = \tfrac{1}{2k}[\sin\, 2(\omega t - kx)]_0^{\lambda} =$$

$$\tfrac{1}{2k}[\sin\, 2(\omega t - k\lambda) - \sin\, 2(\omega t - 0)] =$$

$$\tfrac{1}{2k}[\sin\, 2(\omega t - 2\pi) - \sin\, 2(\omega t - 0)] = 0$$

In the one but last step we have substituted for the wave number $k=2\pi/\lambda$ and then used the periodicity of the function *sin*.

Physics I

9

Thermal physics

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Main Items

- Introduction into thermal physics - heat and temperature
- Thermic and Thermodynamics
  - Units of temperature
  - Thermal expansion of solids, liquids and gases
  - Thermal expansion and expansibility of gases
  - Absolute temperature scale
  - Measurements of temperature
  - Calorimetry – heat capacity, specific and heat
  - Heat conduction
  - Principles of thermodynamics

Introduction into Thermal Physics I

- Thermic deals mainly with the definitions of temperature scale, temperature measurements, thermal effects e.g. expansion, calorimetry and heat transfer.
- Thermodynamics deals mainly with the transfer of heat energy into other forms and its wider theoretical background.
- Temperature represents the thermal status of a system tightly connected with its internal energy.
- Heat is special type of energy which flows when bodies of different temperature are in thermal contact.

Introduction into Thermal Physics II

- The access to thermal physics can be :
  - Phenomenological – only macroscopically measurable quantities are used without dealing with the microscopic structure. Even this way very general conclusions can be made.
  - Atomistic – effects are explained on the basis of the existence of atomic structure and microscopic particles. Measurable quantities are evaluated by statistical averaging of microscopic properties.

Temperature Units I

- Most of macroscopic parameters such as the size, density or conductivity to name just some of them depend on temperature.
- Those for which this dependence is simple and possibly close to linear can be used to measure temperature.
- Let's assume such a measurement can be done and the property is $y(t)$. Then we can define the empiric temperature scale the following way :
  - Certain value $y(0)$ can be attributed the zero temperature
  - Other value $y(n)$ can be attributed temperature n degrees.

Temperature Units II

- Into the dependence in which the zero point is already included : $y(t) = y(0) + kt$

  we insert the second point : $y(n) = y(0) + kn$

  and find the scaling factor :

$$k = \frac{y(n) - y(0)}{n}$$

so finally :

$$y(t) = y(0) + \frac{y(n) - y(0)}{n} t$$

Temperature Units III

- The temperature in using this scale can be found :

$$t = \frac{y(t) - y(0)}{y(n) - y(0)} n$$

- In the Celsius scale :
  - *0°* (degrees) is the temp. of freezing of water
  - *100°* is the temperature of boiling of water

  both at the ambient pressure $1.01325\ 10^5\ Pa$
  - The temperature interval is divided evenly

Temperature Units IV

- Several temperature scales were defined in history and some e.g. the Fahrenheit's are used even now mainly in developing countries and USA. Freezing point of water is attributed 32 °*F* and boiling point of water 212 °*F* so the conversion formulas are :

$$t_F = \tfrac{9}{5} t_C + 32 \quad \text{or} \quad t_C = \tfrac{5}{9}(t_F - 32)$$

- Let's assume temperature can be measured. We shall use the Celsius scale and call the appropriate step Kelvin. Further details of temperature measurements we discuss later.

Thermal Expansion

- Microscopically the thermal expansion can be explained by the asymmetry of the potential well farther from the equilibrium point.
  - at nonzero absolute temperature particles can be at any distance which meets the energy conditions.
  - since the repulsion slope is steeper than the attractive the mean distance of particles grows with temperature. This mean distance is in fact measured macroscopically.

Thermal Expansion of Solids I

- Let's have a rod with a certain length l_0 at some reference temperature, usually $0°$ or $20°\ C$. at small temperature difference the prolongation is proportional to
  - temperature
  - the original length l_0 :

$$\Delta l = l(t) - l_0 = l_0 \alpha t \Rightarrow$$

$$l(t) = l_0 (1 + \alpha t)$$

- The relative prolongation (=strain) is proportional to the temperature :

$$\varepsilon = \frac{\Delta l}{l_0} = \alpha t \quad \text{or} \quad \varepsilon = \frac{l - l_o}{l_o} = \alpha (t - t_0)$$

- $\alpha\ [K^{-1}]$ is the coefficient of linear thermal expansion.

Thermal Expansion of Solids II

- For higher precision or in wider temperature interval quadratic term has to be added :

$$l(t) = l_0(1 + \alpha_1 t + \alpha_2 t^2)$$

α_1 $[K^{-1}]$ is the linear and
α_2 $[K^{-2}]$ the quadratic coefficient of linear thermal expansion.

Thermal Expansion of Solids III

- Cu: $\alpha_1 = 16.7\ 10^{-6}\ K^{-1}$, $\alpha_2 = 0.9\ 10^{-9}\ K^{-2}$
- steel: $\alpha_1 = 12\ 10^{-6}\ K^{-1}$
- glass (Pyrex): $\alpha_1 = 3\ 10^{-6}\ K^{-1}$
- glass : $\alpha_1 = 9\ 10^{-6}\ K^{-1}$
- quartz : $\alpha_1 = 0.5\ 10^{-6}\ K^{-1}$
- special alloy Ni(36)Fe : $\alpha_1 = 0.9\ 10^{-6}\ K^{-1}$
- The order of α_1, describing the main effect, is
  - for metals 10^{-5} K^{-1}
  - for non-metals e.g. ceramic 10^{-6} K^{-1}

Thermal Expansion of Solids IV

- Let's compare thermal expansion : $\varepsilon = \frac{\Delta l}{l_0} = \alpha t$

  with Hook's law : $\varepsilon = \frac{\Delta l}{l_0} = k\sigma = \frac{1}{E}\sigma$

  Obviously temperature load can lead to (large) mechanical stress. This fact has to be taken into account when constructing machines or buildings :

$$k\sigma = \alpha t \Rightarrow \sigma = \frac{\alpha t}{k} = \alpha E t$$

Thermal Expansion of Solids V

- Thermal expansion of volume :
  - For typical values of α their higher powers are negligible so the coefficient of volume thermal expansion β is approximately three times the coefficient of linear thermal expansion α.

$$V(t) = V_0(1+\beta t) = V_0(1+\alpha t)^3 \cong V_0(1+3\alpha t)$$

  - A cavity expands the same way as if it was filled with the material of the walls.

Thermal Expansion of Solids VI

- If the mass doesn't change then the change of density with temperature is :

$$\rho(t) = \frac{m}{V(t)} = \frac{\rho_0 V_0}{V_0(1+\beta t)} \frac{(1-\beta t)}{(1-\beta t)} \Rightarrow$$

$$\rho(t) = \rho_0(1-\beta t)$$

Thermal Expansion of Liquids I

- The temperature expansion of liquids is two orders of magnitude larger an effect and the behavior is more complicated. If we used the same approach as in the case of solids :

$$V(t) = V_0(1 + \beta(t)t)$$

- However, coefficient of volume thermal expansion β (t) depends on the temperature even in the first approximation , so this approach doesn't work.

Thermal Expansion of Liquids II

- More accurate description of thermal expansion needs using the cubic polynomial :

$$V(t) = V_0(at^3 + bt^2 + ct + 1)$$

	a	b	c
Hg:	0	7.8 10^{-9}	1.82 10^{-4}
EtOH:	7.3 10^{-9}	1.85 10^{-6}	7.45 10^{-4}
H_2O:	-6.8 10^{-8}	8.51 10^{-6}	-6.43 10^{-5}

Thermal Expansion of Liquids III

- expansion of mercury is small but almost linear
- expansion of ethanol is large but non-linear
- water exhibits anomalous behavior since its highest density is at *4° C*. Thanks to this fine feature life probably exists on the Earth.
- for $\beta(t)$ and the density ρ we may use formulas :

$$\beta = \frac{1}{V}\frac{\partial V}{\partial t} = \frac{3at^2 + 2bt + c}{at^3 + bt^2 + ct + 1} = \beta(t)$$

$$\rho(t) = \frac{\rho_0}{at^3 + bt^2 + ct + 1}$$

Thermal Expansion of Gases I

- At constant pressure gasses follow the Gay-Lusac law (1802-1808) :

$$V(t) = V_0(1 + \alpha_0 t)$$

- Interestingly for most of diluted gasses the thermal expansion coefficient is the same :

$$\alpha_0 = 0.003661\,K^{-1} \Rightarrow \frac{1}{\alpha_0} = 273.15 \pm 0.002\,K$$

Thermal Expansibility of Gases I

- At constant volume a law analogous to the Gay-Lusac law holds :

$$p(t) = p_0(1+\alpha_0 t)$$

- For most of diluted gasses the thermal expansion coefficient is again :

$$\alpha_0 = 0.003661\,K^{-1} \Rightarrow \frac{1}{\alpha_0} = 273.15 \pm 0.002\,K$$

The Absolute Temperature Scale I

- The fact that thermal coefficients of expansion and expansibility are the same means all isobars and isochors are straight lines which intersect the t-axis at the value $-273.15°\,C$.
- If the zero of the new scale is set at $-273.15°\,C$ and the step is left the same then all linear dependencies become direct proportionalities.
- This is how the absolute scale is defined.

The Absolute Temperature Scale II

- The unit in the absolute temperature scale is 1 Kelvin [K].
- Be aware that the sometimes used term 'degrees Kelvin' is wrong.
- In the absolute scale equations for isochoric and isobaric effect simplify to :

$$\frac{V(T)}{T} = \frac{V_0}{T_0}; \qquad \frac{p(T)}{T} = \frac{p_0}{T_0}$$

The Absolute Temperature Scale III

- Since isochors and isobars pass through the origin only one calibration point is needed.
- As a convention a triple point of water, the point where water exists in all three states of matter exists :

$$T_T = 273.16\ K \approx 0.01°\ C$$

Zeroth Law of Thermodynamics

- Measurements of temperature is based on the zero principle of thermodynamics :
  - If two systems are placed into a thermal contact, they reach sooner or later, the state of thermal equilibrium.
  - If system A is in thermal equilibrium with system B and also with system C, then it can be assumed that systems B and C must be in thermal equilibrium without putting them in thermal contact.
  - The thermal equilibrium of A with B and A with C can be verified at different time or in different place. One verification may be called calibration the other measurement.

Measurements of Temperature I

- The zero principle and some convenient thermal effect can be used for temperature measurements. The effect should mainly :
  - be easy to perform
  - be sensitive
  - be linear or close to linear
  - not influence the measured temperature significantly
- These requirements can't be all matched at the same time so some compromise is made depending on the situation.

Measurements of Temperature II

- A typical example are liquid thermometers :
  - the expansion of the container is negligible
  - mercury behavior is close to linear but the tube must be very thin to enhance the sensitivity.
  - ethanol thermometers are not so linear but they are more sensitive and less dangerous if they break.

Measurements of Temperature III

- The most accurate and most linear but not easy to use is the constant-volume gas thermometer :
  - A bulb filled with diluted gas is connected to two glass pipes joint in their lower part by an u-shape flexible hose filled with mercury.
  - Before the measurement mercury in the pipe next to the bulb reaches a certain reference mark. When temperature in the bulb changes e.g. increases, gas in the bulb expands. This shifts the level of mercury down and the rightmost pipe has to be lifted to place the mercury level back to keep the volume in the container constant.
  - Temperature at constant volume is proportional to the pressure which can be easily measured from the difference of levels in both glass tubes.

Calorimetry I

- It two bodies with different temperatures come into thermal contacts :
  - according to zeroth principle of thermodynamics heat flows between them until they reach thermal equilibrium and be at the same temperature.
  - During this process the originally warmer body looses energy, the colder receives it and the total energy is conserved.

Calorimetry I

- Heat necessary to worm up a body one Kelvin is called its heat capacity [$J\ K^{-1}$].
- For homogeneous materials it makes sense to relate if further to unit of mass. Then it is called specific heat [$J\ kg^{-1}\ K^{-1}$].
- Specific heat is the ability to accumulate heat energy and it is deeply connected with the structure of the matter and generally depends on temperature.

Calorimetry II

- Materials exist in several phases (states).
- A certain change of phase occurs at particular temperature and is connected with exchange of latent heat [J/kg] which is energy that is needed or released for the change of 1 kg of matter. The example is heat of fusion or heat of vaporization.

Calorimetry III

- Heat exchanges e.g. measurements of specific or latent heats are studied in calorimeters. They
  - are well thermally isolated containers
  - their heat capacity must be known or found by a special measurement - calibration

Calorimetry IV

- In the calorimeter with heat capacity K, let's have m_2 of matter with specific heat c_2 and temperature t_2.
- Let's add m_1 of matter with specific heat c_1 and temperature t_1 (e.g.$>t_2$).
- Assume the final temperature is t. The energy is conserved. So the energy lost by the added warmer matter was gained by the calorimeter and the matter it originally contained:

$$m_1 c_1 (t_1 - t) = (K + m_2 c_2)(t - t_2)$$

Heat Conduction I

- The ability to conduct heat depends on
  - the area of contact
  - temperature gradient
  - thickness of the conducting layer
  - heat conductivity k [$W\ m^{-1}\ K^{-1}$] of material
- The power = transferred heat per second is :

$$H = \frac{Q}{t} = kS\frac{T_h - T_l}{L} = \frac{T_h - T_l}{\frac{L}{kS}} = \frac{T_h - T_l}{R}$$

Heat Conduction II

- Geometric and material parameters can be included in new parameter - heat resistance R.
- Then properties of composed layers can be calculated using analogy with electric circuits using the same mathematical tools.
- Let's have a sandwich of two materials and assume the temperature of their interface is T_x . The heat transfer through both must be the same (eq. of continuity) :

$$H = k_1 S_1 \frac{T_h - T_x}{L_1} = k_2 S_2 \frac{T_x - T_l}{L_2}$$

Heat Conduction III

- From equation : $$\frac{T_h - T_x}{R_1} = \frac{T_x - T_l}{R_2}$$
- It follows : $$T_x = \frac{T_h R_2 + T_l R_1}{R_1 + R_2}$$
- And so : $$H = \frac{T_h - T_l}{R_1 + R_2} = \frac{T_h - T_l}{R_{12}}$$
- Where : $R_{12} = R_1 + R_2$ so heat resistance adds as for serial combination of resistors

Heat Conduction IV

- This analogy can be generalized for sandwich of more layers.
- This analogy with serial combinations of resistors stems from the fact that the power transferred through each layer must be the same as is the case of current in each serially connected resistors.
- If we changed the succession of the layers the temperatures at the interfaces would change but the total heat resistance as well as the transferred power would remain the same.

Introduction into Thermodynamics I

- Thermodynamics deals with changes thermal energy from or to other types.
- We deal with a system which is a certain way separated from the environment :
  - closed system doesn't exchange particles
  - isolated system doesn't exchange heat
- We deal with systems in equilibrium.

Introduction into Thermodynamics II

- The state of a system (in equilibrium) is described by parameters which are divided to extensive or intensive.
  - If A is an intensive parameter it's not dependent on the size of the system :

$$\lim_{V \to \infty} A = konst.$$

Intensive parameters are e.g. pressure, temperature and all specific and molar quantities.

Introduction into Thermodynamics III

- If a parameter B is extensive then :

$$\lim_{V\to\infty} B = \infty \wedge \lim_{V\to\infty} \frac{B}{V} = konst.$$

Extensive parameters are e.g. volume, internal energy, and all thermodynamic potentials entropy, enthalpy, Gibbs and Helmholtz free energy.

Introduction into Thermodynamics IV

- In a system processes take place. They can :
  - start from some initial state and finish in some end state or they can be loops.
  - be either reversible or irreversible. In fact all real processes are irreversible. Reversible processes would have to proceed very slowly so that the system is (almost) in equilibrium all the time and these processes can proceed in both directions.

1st law of thermodynamics I

- Energy can be input into the system in the form of :
  - Heat. Heat dQ input into the system is considered positive.
  - Work. Work dA done on the system is considered positive.
  - For the volume work done on the system it can be shown that :

$$dA = -Fdx = -pSdx = -pdV$$

1st law of thermodynamics II

- 1st law of thermodynamic says that the energy of a system is conserved :

$$\bar{d}A + \bar{d}Q = dU$$

  - Energy input into the system in the form of mechanical work A or as heat Q leads to the increase of the internal energy U.
  - While $\bar{d}Q$ and $\bar{d}A$ depend on the path their sum dU does not. It is a thermodynamic potential.

2^{nd} law of thermodynamics I

- Experiment shows that processes have their natural direction leading to higher stability of the system. They would never proceed in reverse direction even if the law of energy conservation would not be violated.
- That means that other quantities and rules must exist which describe this fact.

2^{nd} law of thermodynamics II

- R. Clausius : heat can't flow from a colder body to a warmer one without the input of work.
- W. Thompson: A process that would change continuously heat taken from one thermostat to mechanical work without compensation - giving part of heat to a colder body, doesn't exist. That would be a perpetum mobile of the second kind.

2^{nd} law of thermodynamics III

- Both formulations are equivalent :
  - A process that could violate the R.C. could also remove compensation so W.T. would not hold.
  - A process that could violate W.T. could pump heat from a thermostat, change it to work and then e.g. by friction to heat that could flow to warmer thermostat. So also R.C. would not hold.

Entropy I

- It can be shown that for the reversibly exchanged heat the ratio $\frac{dQ}{T}$ state variable. If we imagine a loop divided into many small Carnot cycles :

$$\sum \frac{dQ}{T} = 0 \Rightarrow \oint \frac{dQ}{T} = 0$$

- The last term is called the Clausius' integral for reversible processes.

Entropy II

- Using the analogy with definition of conservative fields it is easy to show that the change of the ratio doesn't depend on path so it is a thermodynamic potential - state function - entropy :

$$\int_{I:a}^{b} \frac{dQ}{T} + \int_{II:b}^{a} \frac{dQ}{T} = 0 \Rightarrow \int_{I:a}^{b} \frac{dQ}{T} = \int_{II:a}^{b} \frac{dQ}{T}$$

- The entropy S is a state function which describes the direction or reversibility of processes.

Irreversible Heat Engine

- The effectivity of irreversible process will always be smaller than the effectivity of reversible process :

$$\eta \equiv \frac{A'}{Q_1} = \frac{Q_1 + Q_2}{Q_1} < \frac{T_1 - T_2}{T_1}$$

- This corresponds to smaller work done and thereby bigger compensation :

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} < 0 \Rightarrow \sum \frac{dQ}{T} < 0 \Rightarrow \oint \frac{dQ}{T} < 0$$

Entropy III

- The Clausius' integral can then be generalized for all processes :

$$\oint \frac{dQ}{T} \leq 0$$

- The equality sign characterizes reversible and inequality irreversible processes where entropy is generated and leaves the system during compensation.
- General relation between the entropy dS and $\frac{dQ}{T}$ can be found is we proceed from the state *1* to *2* irreversibly [*I*] and back from *2* to *1* reversibly [*R*].

Entropy IV

- The total cycle is irreversible so :

$$\int_{1I}^{2}\frac{dQ}{T}+\int_{2R}^{1}\frac{dQ}{T}=\int_{1I}^{2}\frac{dQ}{T}+\int_{2R}^{1}dS=\int_{1I}^{2}\frac{dQ}{T}-\int_{1R}^{2}dS<0$$

- So for all processes :

$$dS \geq \frac{dQ}{T}$$

as before equality is valid for reversible processes.

Entropy V

- Since all real processes are irreversible the entropy always grows.
- We show that this roughly means directing to lower order, more precisely to the state that is accomplished by more microstates or quantum states.

Example - Heat conduction I

A piece of Cu $m_1 = 10\ g$ and $T_1 = 350\ K$ is put into the thermal contact with identical piece which is at $T_2 = 210\ K$. The specific heat is $c = 389\ kJ\ kg^{-1}K^{-1}$.

What will be the final temperature?

What total energy will be transferred?

What is the entropy change when the first 0.1 J is transferred?

From the energy conservation :

$\Delta U = m_1 c(T_1 - T) = m_2 c(T - T_2)$.

So : $T = 320\ K\ a\ \Delta U = 11.7\ J$.

Example - Heat conduction II

When exchanging the first $0.1\ J$ it can be assumed that temperatures of both bodies have the original values. The entropy of the warmer body decreases by $\Delta S_1 = -\ 0.1/350 = -2.86\ 10^{-4}\ JK^{-1}$. When the heat is given to the colder body its entropy increases by $\Delta S_2 = \ 0.1/290 = 3.45\ 10^{-4}\ JK^{-1}$. So totally the entropy increases by :

$$\Delta S = \Delta S_1 + \Delta S_2 = 0.59\ 10^{-4}\ JK^{-1}$$

Statistical Meaning of Entropy I

- Entropy shows the direction of processes and sets criteria of their reversibility.
- We have shown general definition of entropy and its behavior in reversible or irreversible processes. It is precise but not particularly illustrative. We try a different approach to improve this.

Statistical Meaning of Entropy II

- Statistical approach can considerably improve understanding the meaning of the entropy :
  - We illustrate that the state of a system can change by itself only in the direction leading to the more probable state, the one that is accomplished by more micro-states or in other words had higher thermodynamic weight.
  - The equilibrium is the most probable state.
  - At non-zero absolute temperature the state of a system fluctuates through the states close to the equilibrium.

Statistical Meaning of Entropy III

- Let's illustrate the meaning of entropy using an example :
  - it is simplified since it is using only the configuration space
  - in reality the entropy depends also on velocities of particles and it has to be described fully only in the phase space

Statistical Meaning of Entropy IV

- Let's have a playground divided in two halves by a line on which 4 players with numbers on their back chaotically move = an analogue to 4 particles of ideal gas in a container.
- The state (=macro-state) of this system is what we see from far above wherefrom the numbers can't be read and players can't be distinguished. The state can be characterized by the number of players who are in the left half of the field.
- Various macro-states can be distinguished by measurements.
- The state when there is e.g. one player in the left half of the field can be accomplished by four micro-states since it can be either player 1 or 2 or 3 or 4.
- Various micro-states can't be distinguished by measurements.

Statistical Meaning of Entropy V

- During chaotic movement of players the micro-states will perpetually change and appearance of any of them will have the same probability.
- The thermodynamic weight Ω is the total number of micro-states which a certain system can totally have under current conditions.
- The probability of a certain (macro-)state is then equal to the number of micro-states by which is this state is accomplished divided by the thermodynamic weight.
- Let's look at it in a deeper detail :

Statistical Meaning of Entropy VI

- Our system has totally 5 states distinguished by number of players in the left half.

state	nr of micro-states	probability:
0	1	0.0625
1	4	0.25
2	6	0.376
3	4	0.25
4	1	0.0625

- The configuration state than has *16* micro-states.

Statistical Meaning of Entropy VII

- If we return to particles in a container, we see that even in this case of very small number of particles the states have considerably different probabilities.
- In our case of only 4 particles the first state is six times less probable than the most probable state.
- In a container divided in two halves every particle generally generates two states since it can be either in the left or the right half. If there is n particles in the container there will be totally 2^n micro-states.
- The state when there is m particles is in the left half is accomplished by $\dfrac{n!}{m!(n-m)!}$ micro-states.

Statistical Meaning of Entropy VIII

- The ratio of probability of bizzare states to the states near to the equilibrium is astronomically low even for very small n.
- Even the state when the left half is empty is n-times less probable than when there is just one particle. In the case of macroscopic systems n ~ N_A.

Statistical Meaning of Entropy IX

- Let's check this for $n = 20$ $(2^{20} = 1048576)$:

state	nr micro-states	probability	sym-sum
0	1	$9.5\ 10^{-7}$	
1	20	$1.9\ 10^{-5}$	
5	15504	0.015	(5-15) 0.988
6	38760	0.037	(6-14) 0.959
7	77520	0.074	(7-13) 0.885
8	125970	0.120	(8-12) 0.737
9	167960	0.160	(9-11) 0.497
10	184756	0.176	

Statistical Meaning of Entropy X

- Direct calculation of the formula $\frac{n!}{m!(n-m)!}$ fails already for *n ~ 180*.
- The problem is that numbers used in calculations are much large than the result. For large n it can be circumvented by using logarithm of the formula and then the Stirling equation :

$$\ln(n!) \cong n.\ln(n) - n + 1 \cong n.\ln(n)$$

- So : $n.\ln(n) - m.\ln(m)n - (n-m).\ln(n-m)$

Statistical Meaning of Entropy XI

- For large number of particles the probable states are near to the equilibrium state called also the central configuration.
- The probabilities of these states are similar so the system fluctuates around the central state.
- If we separate the container by a diaphragm we can start from some improbable state e.g. that all particles are just in one half. After removing the diaphragm it is clear that the system moves towards the most probable central state.

Statistical Meaning of Entropy XII

- The entropy depends on the thermodynamic weight Ω but it must be an extensive quantity.
- If we join two subsystems this must hold : $\Omega_{AB} = \Omega_A \,.\, \Omega_B$ a $S_{AB} = S_A + S_B$
- Ludwig Boltzman (1844-1906) had found :

$$S = k \ln \Omega$$

$k = (1.3806503 \pm 0.0000024)\ 10^{-23}\ J\ K^{-1}$ is the Boltzman constant.

Macro-state and Measurable quantities I

- Let's assume the isolated system in equilibrium can be in macro-states $i=1\ldots n$
  - each macro-state is accomplished by Ω_i micro-states e.g. the probability $P_i = \Omega_i/\Omega$, where Ω is the total number of micro-states and some value y_i of measurable quantity corresponds to it
  - The mean value of the measurable quantity y is:

$$< y > = \sum P_i y_i = \tfrac{1}{\Omega} \sum \Omega_i y_i$$

The number of micro-states

n particles can be organized in a row by *n!* ways :

The first particle can be at any of the *n* positions. The second particle can then be in any of the rest *n-1* positions. The third in *n-2* positions and so on.

The state when there is *m* particles in the left part and *n-m* particles in the right one we have to divide *n!* by all the possibilities *m* particles and n-m particles can be organized since they don’t' change the macro-state. The this macro-state is realized by:

$$\frac{n!}{m!(n-m)!}$$

micro-states.

Near the Central Configuration I

The central configuration is the most probable state that is also realized by the highest number of micro-states.

In the case of a container divided in halves it is $N/2{:}N/2$. Similarly in the sub-volume $v = V/10$ the most probable number of particles is $N/10$ and $9N/10$ particles in the rest.

We study several particle distributions and see that with the growing number of particles the central peak relatively grows but the values near the maximum become similar.

Near the Central Configuration II

Let's have *N* particles in a container that is divided into two halves. We find the probability *p(cc)* of the central configuration. Let's assign a state near the central configuration by a asymmetry parameter *s*: In this state there is *N/2-s* in the left half and *N/2+s* in the right half. Let's find the probability *p(10%)* of the state where the asymmetry is 10%. Finally we find *I*(98%) the number of states <n/2-s, n/2+s>, in which there is totally 98% micro-states :

N	p(ck)	p(10%)	I(98%)
10	0.246	0.2015	4
20	0.176	0.1201	6
50	0.112	0.0419	9
100	0.080	0.0108	13

Near the Central Configuration III

Even from this short table the trends for growing N are clearly visible :

1) The probability of the central configuration decreases

2) The probability of states with the same distance from the central configuration also decreases.

3) The interval around the central configuration where the system stays with a certain probability absolutely grows.

Near the Central Configuration IV

For large number of particles ~N_A this means the system passes all the time through great number of micro-states near to the central configuration but 'near' may mean that in the left half there can be several millions of particles less than in the right. Since several millions <<< N_A this asymmetric state is not so asymmetric and its probability is close to the probability of the central configuration.

Quantities that depend on the number of particles fluctuate accordingly.

If the number of particles is very small this doesn't have to be the case.

Physics

10

Electrostatics I

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Main Topics

- Electric Charge and its Properties
  - Coulomb’s Law
  - Electric Field and Electric Intensity
- The Gauss’ Law
  - The Electric Flux
  - The Charge Density
  - Using Gauss Law
- Conservative Fields
  - The Existence of the Electric Potential
  - Relations of the Potential and Intensity
  - The Gradient
  - Electric Field Lines and Equipotential Surfaces.
  - Motion of Charged Particles in Electrostatic Fields

I-1 Electric Charge

Why Electrostatics?

- Many important properties of the Nature exist due to electric interactions of charged particles.
- We shall first deal with fields and charges which are static = do not move.
- It is for simplicity but such fields really exist, if some equilibrium can be reached!

Demonstration of Electrostatic Effects

- A comb after it has been run through hair attracts little pieces of paper. The force is a long-distance one. It can be attractive or repulsive.
- We attribute these forces to the existence of a property we call the electric charge.
- Bodies can be charged by conduction via contact with other bodies but even remotely by induction.
- Using some materials we can easily discharge charged bodies, these are conductors. By others it is slow or even impossible, they are insulators

Main Properties of Charges

- Since both the attractive and the repulsive forces exist, charges must be of two kinds, positive and negative. Unlike charges attract and like charges repel themselves.
- Charges are quantized. They can only be isolated in integer multiples of the elementary charge

$$e = 1.602\ 10^{-19}\ \mathrm{C}$$

- In all known processes charges appear or disappear only in pairs (+q and -q), so the total charge is conserved
- Charge is invariant to the Lorentz transformation

Main Properties of Electrostatic Interactions

- Charged particles act by a force on each other. Forces :
  - are Long-distance – mediated by electric field
  - obey the principle of superposition
- Mutual interaction of two static point charges is described by Coulomb's Law

Coulomb’s Law I

- Let us have two point charges Q_1 and Q_2 at the distance r apart. Then the magnitude of the interaction force is:

$$F = k\, Q_1\, Q_2 / r^2$$

  - The SI unit of charge is 1 Coulomb [1 C]
  - $k = 9\ 10^9$ Nm²/C²
  - $k = 1/4\pi\varepsilon_0$
  - $\varepsilon_0 = 8.85\ 10^{-12}$ C^2/Nm^2 is the permitivity of vacuum

Coulom’s Law II

- Since forces are involved the directional information is as important as the magnitude.
- To get the full information let’s place Q_1 into the origin and let describe the position of Q_2 by the radius vector $\vec{r}$. Then the force acting on Q_2 is :

$$\vec{F}_{21}(\vec{r}) = \frac{kQ_1Q_2}{r^2}\frac{\vec{r}}{r} = \frac{kQ_1Q_2}{r^2}\vec{r}^{\,0}$$

  - Forces act in the straight line joining the charges .
  - Positive force is repulsive .
  - Forces acting on Q_1 and Q_2 are action and reaction of each other .

Coulomb's Law III

- The most general formula we get if we define the position of each charge Q_i ($i=1, 2$) by its own radius vector $\vec{r}_i$. Then the force acting on Q_2 is :

$$\vec{F}_{21}(\vec{r}_2) = \frac{kQ_1Q_2(\vec{r}_2 - \vec{r}_1)}{|\,\vec{r}_2 - \vec{r}_1\,|^3}$$

  - Since the force depends only on the difference between the radius vectors, the position of the origin is arbitrary .

Comparison with the Force of Gravity

- Formally, Coulomb's Law is Analogous to Newton's Gravitational Law
  - But electrostatic force is ~ 10^{42} (!) times stronger
  - Such a weak force still dominates the universe because matter is usually neutral
  - Charging something means to break very slightly the great equilibrium

The Concept of the Field

- If a charge is located at some point in the space it sends around an information about its position, polarity and magnitude. The information spreads with the speed of light. It can be “received” by another charge . The interaction of a charge with field produces a force.

Electric Intensity I

- Electrostatic field could be described by taking some test charge Q and recording the vector of the force $\vec{F}(\vec{r})$ acting on it in every point of interest.
- This description would, however, depend on the magnitude and polarity of the test charge and these properties would have to be provided as the additional information to make the description unique.

Electric Intensity II

- By dividing of the force by the magnitude of the 'test' charge the field acts on the electric intensity of this field is defined :

$$\vec{E}(\vec{r}) = \frac{\vec{F}(\vec{r})}{Q}$$

- It is a unique property of the described field, now.
- Numerically it is equal to the force which would act in the particular point on a unit positive charge, but be careful with dimensions.

Electric Intensity III

- It is important to realize that by dividing by the magnitude of the charge, the information, how the charge ‘feels’ the field, becomes an objective property of the field.
- The same field acts on various charges by various forces which can be even opposite due to the existence of two polarities of charges.

Electric Field Lines I

- Electric field is a three dimensional vector field which is in general case difficult to visualize.
- In cases of simple symmetry, it is possible to use electric field lines which are lines tangent to vectors of the electric intensity in every point. The magnitude of the field can be illustrated by their length or density.

Electric Field Lines II

- A positive charge of a very small mass would move along a certain field line in the direction of the electric field while negative charge would move in the opposite direction.
- Field lines can't cross!

I-2 Gauss’ Law

The Electric Flux

- The electric flux is defined as

$$d\Phi_e = \vec{E} \cdot d\vec{A}$$

  It represents amount of electric intensity $\vec{E}$ which flows perpendicularly through a surface, characterized by its outer normal vector $d\vec{A}$. The surface must be so small that $\vec{E}$ can be considered constant there.
- Let's revisit the scalar product.

The Gauss' Law I

- Total electric flux through a closed surface is equal to the net charge contained in the volume surrounded by the surface divided by the permitivity of vacuum .

$$\oint d\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{\sum Q}{\varepsilon_0}$$

- It is equivalent to the statement that field lines begin in positive charges and end in negative charges.

The Gauss' Law II

- Field lines can both begin or end in the infinity.
- G. L. is roughly valid because the decrease of intensity the with r^2 in the flux is compensated by the increase with r^2 of surface of the sphere.
- The scalar product takes care of the mutual orientation of the surface and the intensity.

The Gauss' Law III

- If there is no charge in the volume each field line which enters it must also leave it.
- If there is a positive charge in the volume then more lines leave it than enter it.
- If there is a negative charge in the volume then more lines enter it than leave it.
- Positive charges are sources and negative are sinks of the field.
- Infinity can be either source or sink of the field.

The Gauss' Law IV

- Gauss' law can be taken as the basis of electrostatics as well as Coulomb's law. It is actually more general!
- Gauss' law is useful:
  - for theoretical purposes
  - in cases of a special symmetry

The Charge Density

- In real situations we often do not deal with point charges but rather with charged bodies with macroscopic dimensions.
- Then it is usually convenient to define the charge density i.e. charge per unit volume or surface or length, according to the symmetry of the problem.
- Since charge density may depend on the position, its use makes sense mainly if the bodies are uniformly charged e.g. conductors in equilibrium.

A Point Charge I

- As a Gaussian surface we choose a spherical surface centered on the charge.
- Intensity $\vec{E}$ is perpendicular to the spherical surface in every point and so parallel (or antiparallel) to its normal.
- At the same time *E* is constant on the surface, so :

$$\oint \vec{E} \cdot d\vec{A} = E \oint dA = E 4\pi r^2 = \frac{q}{\varepsilon_0}$$

A Point Charge II

- So we get the same expression for the intensity as from the Coulomb's Law :

$$E(r) = \frac{q}{4\pi\varepsilon_0 r^2}$$

- Here we also see where from the "strange term" $\frac{1}{4\pi\varepsilon_0}$ appears in the Coulomb's Law!

An Infinite Uniformly Charged Wire I

- Conductive wire (in equilibrium) must be charged uniformly so we can define the length charge density as charge per unit length:

$$\tau = \frac{Q}{L} \quad [Cm^{-1}]$$

- Both Q and L can be infinite, yet have a finite ratio.
- The wire is axis of the symmetry of the problem.

An Infinite Uniformly Charged Wire II

- Intensity lies in planes perpendicular to the wire and it is radial.
- As a Gaussian surface we choose a cylindrical surface (of some length L) centered on the wire.
- Intensity $\vec{E}$ is perpendicular to the surface in every point and so parallel to its normal.
- At the same time E is constant everywhere on this surface.

Infinite Wire III

- Flux through the flat caps is zero since here the intensity is perpendicular to the normal.
- So :

$$\oint \vec{E}\cdot d\vec{A} = E\oint dA = E2\pi rL = \frac{\tau L}{\varepsilon_0}$$

$$E(r) = \frac{\tau}{2\pi r\varepsilon_0}$$

Infinite Wire IV

- By making one dimension infinite the intensity decreases ~ $1/r$ instead of $1/r^2$ which was the case of a point charge!
- Again, we can obtain the same result using the Coulomb's law and the superposition principle but it is "a little" more difficult!

An Infinite Charged Conductive Plane I

- If the charging is uniform, we can define the surface charge density :

$$\sigma = \frac{Q}{A} \qquad [Cm^{-2}]$$

- Again both Q and A can be infinite yet reach a finite ratio, which is the charge per unit surface.
- From the symmetry the intensity must be everywhere perpendicular to the surface.

Infinite Plane II

- As a Gaussian surface we can take e.g. a cylinder whose axis is perpendicular to the plane. It should be cut in halves by the plane.
- Nonzero flux will flow only through both flat cups (with some magnitude A) since $\vec{E}$ is perpendicular to them.

$$\oint \vec{E} \cdot d\vec{A} = E \oint dA = E 2A = \frac{\sigma A}{\varepsilon_0}$$

$$E(r) = \frac{\sigma}{2\varepsilon_0} = E$$

Infinite Plane III

- This time $\vec{E}$ doesn't change with the distance from the plane. Such a field is called homogeneous or uniform!
- Note that both magnitude and direction of the vectors must be the same if the vector field should be uniform.

Quiz: Two Parallel Planes

- Two large parallel planes are d apart. One is charged with a charge density σ, the other with $-\sigma$. Let E_b be the intensity between and E_o outside of the planes. What is true?
  - A) $E_b = 0$, $E_o = \sigma/\varepsilon_0$
  - B) $E_b = \sigma/\varepsilon_0$, $E_o = 0$
  - C) $E_b = \sigma/\varepsilon_0$, $E_o = \sigma/2\varepsilon_0$

I-3 Electric Potential

Conservative Fields

- There are special fields in the Nature in which the total work done when moving a particle on along any closed path is zero. We call them conservative.
- Such fields are for instance:
  - Gravitational - we move a massive particle
  - Electrostatic - we move a charged particle

The Existence of the Electric Potential

- From the definition of a conservative field it can be shown that work done by moving a charged particle from some point *A* to some other point *B* doesn’t depend on the path but only on the difference of some scalar quality in both points. This quality is called the electric potential φ.

Work Done on Charge in Electrostatic Field by an External Agent I

- If we (as an external agent) move a charge q from some point A to some point B then we do by definition work :

$$W(A\text{->}B) \equiv q[\varphi(B)-\varphi(A)]$$

Work Done on Charge in Electrostatic Field II

- Since doing positive work on some particle means increasing its energy, we can define a potential energy U

$$U = q\varphi$$

- This definition clearly matches the above :

$$W(A\text{->}B) = q[\varphi(B) - \varphi(A)] = U(B) - U(A)$$

Work Done on Charge in Electrostatic Field III

- In almost all situations we are interested in the difference of two potentials. We define this difference as the voltage V

$$V_{AB} = \varphi(B) - \varphi(A)$$

- Then:

$$W(A\text{->}B) = q\, V_{AB}$$

Work Done on Charge in Electrostatic Field IV

- So we come to the general formula:

$$W=q[\varphi(B)-\varphi(A)]=U(B)-U(A)=qV_{AB}$$

- Try to understand well the difference
  - between the potential φ, the potential energy U and the voltage V!
  - between the work done by the field W' and done by an external agent $W = -W'$ (skiing)!

The Impact of the Potential

- Since the potential exists, we can describe the electrostatic field fully using the scalar potential field instead of the vector intensity field :

$$\vec{E}(\vec{r}) \rightarrow \varphi(\vec{r})$$

  - We need only one third of information
  - Superposition means just adding numbers
  - Some terms converge better

Relations Between Potential and Intensity I

- It is convenient to study this relation first in terms of potential energy and force so we don't have to care about the polarity of the charge and we can use examples from the gravitation field.
- Let's have a charged particle in a field which acts by a force $\vec{F}$ on it.
- If the particle moves by $d\vec{l}$ the field does work *W'* on it :

$$dW' = \vec{F} \cdot d\vec{l}$$

Relations of φ versus $\vec{E}$ II

- The sign of this work depends on the projection of the path vector $d\vec{l}$ into the vector of force $\vec{F}$.
- If it has the same direction as the force, the field does a positive work. Such a shift can take place without some external agent acting. But the it must be done at the cost of lowering the potential energy of the particle:

$$dW' = \vec{F} \cdot d\vec{l} = -dU$$

- Further we can talk only about shifts in the direction of the force or those opposite to it.

Relations of φ versus $\vec{E}$ III

- When shifting the particle into the direction of the force the (positive) work is done by the field and when shifting it into the opposite direction the work is done by an external agent :
  - in this case the potential energy of the particle increases
  - the field can return this energy later
  - that's why we call it potential energy

Relations of φ versus $\vec{E}$ IV

- The work done by the field for a certain path A->B we can get by integration :

$$W'(A \to B) = \int_A^B \vec{F} \cdot d\vec{l} = -[U(B) - U(A)]$$

- Finally, after dividing by the charge we get the relation between the intensity and the potential, we were looking for :

$$\int_A^B \vec{E} \cdot d\vec{l} = -[\varphi(B) - \varphi(A)]$$

Relations of φ versus $\vec{E}$ V

- Let's have a particle with a unit positive charge. Force acting on it is numerically equal to the intensity and its potential energy is numerically equal to the potential in the particular point.
- But we have to understand that
  - the intensity and the potential are properties of the field
  - the force and the potential energy are the properties of the particular particle in the field and their dimensions differ by the factor Q [C].

Relations of φ versus $\vec{E}$ VI

- Let's shift a unit charge ($1C$) in the direction of the intensity by $d\vec{l}$. Then we have:

$$\vec{E} \cdot d\vec{l} = E\,dl = -[\varphi(B) - \varphi(A)] > 0$$

- So : $\varphi(B) = \varphi(A) - Edl$ and the potential decreases in the direction of the intensity and also along the field lines.

Relations of φ versus $\vec{E}$ VII

- So when moving along a field line, we can get the intensity as a change of the potential :

$$E = \frac{-[\varphi(B) - \varphi(A)]}{dl} = -\frac{d\varphi}{dl}$$

- We see that the potential is connected to the integral properties of the intensity while the intensity on the other hand to the derivative properties of the potential.

Uniform ≡ Homogeneous Field I

- The simplest electrostatic field is the uniform or homogeneous field whose intensities are constant vectors (they have the same magnitudes and directions) in every point.
- In a uniform field we can illustrate the properties derived in the easiest way.
- The potential changes only in the direction of the intensities. And it is the only important direction.
- The field lines are all parallel lines.

Uniform ≡ Homogeneous Field II

- Everything derived above is valid now, even for a shift of any distance d along a field line :

$$E = \frac{-[\varphi(B) - \varphi(A)]}{d} = -\frac{\Delta\varphi}{d}$$

- The intensity can be understood as a slope of the change of potential along a field line.

Homogeneous Field III

- If we want to find the work necessary to shift a charge from one point into another one, we have to find what is the projection into the direction of a field lines and we have to take into account what charge we particularly shift.
  - Large charge feels steeper slope of its potential energy than a small one.
  - Negative charge feels the decrease of potential of the field as an increase of its potential energy.

The Units

- The unit of φ or V is 1 Volt.
- $[\varphi] = [U/q] \Rightarrow V = J/C$
- $[E] = [\varphi/d] = V/m$
- $[\varphi] = [kq/r] = V \Rightarrow [k] = Vm/C \Rightarrow$

  $[\varepsilon_0]=CV^{-1}m^{-1}$

I-4 Simple Electrostatic Fields

A Spherically Symmetric Field I

- A spherically symmetric field e.g. a field of a point charge is another important field where the relation between φ and E can easily be calculated.
- Let's have a single point charge Q in the origin. We already know that the field lines are radial and have a spherical symmetry:

$$\vec{E}(\vec{r}) = \frac{kQ}{r^2}\vec{r}^{\,0}$$

A Spherically Symmetric Field II

- The magnitude of E depends only on r

$$E(r) = \frac{kQ}{r^2}$$

- Let's move a "test" charge q equal to unity from some point A to another point B. We study directly the potential! Its change actually depends only on changes of the radius. This is because during the shifts at a constant radius work is not done.

A Spherically Symmetric Field III

- The conclusion: potential φ of a spherically symmetric field depends only on r and it decreases as 1/r

$$\varphi(r) = \frac{kQ}{r}$$

- If we move a non-unity charge q we have again to deal with its potential energy

$$U(r) = \frac{kqQ}{r}$$

The General Formula of $\vec{E}(\varphi)$

- The general formula is very simple

$$\vec{E}(\vec{r}) = -grad\ \varphi(\vec{r})$$

- Gradient of a scalar function $f(\vec{r})$ in some point is a vector :
  - It points to the direction of the fastest growth of the function *f*.
  - Its magnitude is equal to the change of the function *f*, if we move a unit length into this particular direction.

$\vec{E}(\varphi)$ in Uniform Fields

- In a uniform field the potential can change only in the direction along the field lines. If we identify this direction with the *x*-axis of our coordinate system the general formula simplifies to:

$$E(\vec{r}) = -\frac{d\varphi(\vec{r})}{dx}$$

$$F(\vec{r}) = -\frac{dU(\vec{r})}{dx}$$

$\vec{E}(\varphi)$ in Centrosymmetric Fields

- When the field has a spherical symmetry the general formulas simplify to:

$$E(\vec{r}) = -\frac{d\varphi(\vec{r})}{dr} \quad \text{and} \quad F(\vec{r}) = -\frac{dU(\vec{r})}{dr}$$

- This can for instance be used to illustrate the general shape of potential energy and its impact to forces between particles in matter.

The Equipotential Surfaces

- Equipotential surfaces are surfaces on which the potential is constant.
- If a charged particle moves on a equipotential surface the work done by the field as well as by the external agent is zero. This is possible only in the direction perpendicular to the field lines.

Equipotentials and the Field Lines

- We can visualize every electric field by a set of equipotential surfaces and field lines.
  - In uniform fields equipotentials are planes perpendicular to the field lines.
  - In spherically symmetric fields equipotentials are spherical surfaces centered on the center of symmetry.
  - Real and imaginary parts of an ordinary complex function has the same relations.

Motion of Charged Particles in Electrostatic Fields I

- Free charged particles tend to move along the field lines in the direction in which their potential energy decreases.
- From the second Newton's law:

$$\frac{d\vec{p}}{dt} = q\vec{E}$$

- In non-relativistic case:

$$m\vec{a} = q\vec{E} \Rightarrow \vec{a} = \frac{q}{m}\vec{E}$$

Motion of Charged Particles in Electrostatic Fields II

- The ratio q/m, called the specific charge is an important property of a particle.
  1. electron, positron $|q/m| = 1.76\ 10^{11}\ C/kg$
  2. proton, antiproton $|q/m| = 9.58\ 10^{7}\ C/kg$ (1836 x)
  3. α-particle (He core) $|q/m| = 4.79\ 10^{7}\ C/kg$ (2 x)
  4. other ions …
- Accelerations of elementary particles can be enormous!

Motion of Charged Particles in Electrostatic Fields III

- Either the force or the energetic approach is employed.
- Usually, the energetic approach is more convenient. It uses the law of conservation of energy and takes the advantage of the existence of the potential energy.

Motion IV – Energetic Approach

- If in the electrostatic field a free charged particle is at a certain time in a point *A* and after some time we find it in a point *B* and work has not been done on it by an external agent, then the total energy in both points must be the same, regardless of the time, path and complexity of the field :

$$E_{KA} + U_A = E_{KB} + U_B$$

Motion V – Energetic Approach

- We can also say that changes in potential energy must be compensated by changes in kinetic energy and vice versa :
- $(E_{kB} - E_{kA}) + (U_B - U_A) = \Delta E_k + \Delta U = 0$
- $\Delta E_k + q(\varphi_B - \varphi_A) = \Delta E_k + q\Delta\varphi = 0$
- $\Delta E_k + q(\varphi_B - \varphi_A) = \Delta E_k + qV_{AB} = 0$
- In high energy physics 1eV is used as a unit of energy 1eV = 1.6 10^{-19} J.

Motion of Charged Particles in Electrostatic Fields II

- It is simple to calculate the gain in kinetic energy of accelerated particles from :

$$\Delta E_k = -qV_{AB}$$

- When accelerating electrons by few tens of volts we can neglect the original speed.
- But relativistic speeds can be reached at easily reached voltages!

One electron and one proton 0.53 10^{-10} m apart

This corresponds to their distance in hydrogen atom.

$$F_e = -9 \cdot 10^9 (1.6 \cdot 10^{-19} / 0.53 \cdot 10^{-10})^2$$

$$= -8.2 \cdot 10^{-8}\ N$$

Force of this magnitude can be in principle measured macroscopically! This is the secret why matter holds together.

Let us separate protons and electrons from one gram of H and put each group on each pole of the Earth

1 g is 1 gram-molecule of H, so we have N_A=6.02 10^{23} of both types of particles.

$$F_e = -9\cdot 10^9 (1.6\cdot 10^{-19}\, 6.02\cdot 10^{23} / 12.7\cdot 10^6)^2$$

$$= -5.2\cdot 10^5\ N\ (!)$$

Two 1g iron spheres, 1 m apart are attracted by the force of 10 N. What is their excess charge compared to the total charge?

The excess charge:

$$q = \sqrt{10/9\cdot 10^{9}} = 3.3\cdot 10^{-5}\,C \cong 2\cdot 10^{14}\,e$$

The total charge:

$$q_t = \frac{6.02\cdot 10^{23}\cdot 26}{55.8} = 2.8\cdot 10^{23}\,e$$

$$q/q_t \approx 10^{-9}\ (!)$$

Two electrons 1 *m* apart

They are repelled by electric force but attracted by the force of gravitation. Which force will prevail?

$$F_e = 9\cdot 10^{9}(-1.6\cdot 10^{-19})^2 = 2.31\cdot 10^{-28}\ N$$

$$F_g = 6.67\cdot 10^{-11}(9.1\cdot 10^{-31})^2 = 5.54\cdot 10^{-71}\ N$$

$$F_e / F_g = 4.2\cdot 10^{42}\ !!!$$

The scalar ≡ dot product

$$Let\ c = \vec{a} \cdot \vec{b}$$

Definition I. (components)

$$c = \sum_{i=1}^{3} a_i b_i$$

Definition II. (projection)

$$c = |\vec{a}|\left|\vec{b}\right| \cos\varphi$$

Can you proof their equivalence?

Gauss' Law

The exact definition:

$$\oint d\Phi_e = \oint \vec{E} \cdot d\vec{A} = \frac{\sum q}{\varepsilon_0}$$

In cases of a special symmetry we can find Gaussian surface on which the magnitude E is constant and $\vec{E}$ is everywhere parallel to the surface normal. Then simply:

$$\oint d\Theta_e = E\ A = \frac{\sum q}{\varepsilon_0}$$

Infinite Wire by C.L.– die hard!

Only radial component E_r of $\vec{E}$ is non-zero

$$E_r = E\sin\varphi;\ \rho = \frac{r}{\sin\varphi};\ x = -r\,arctg\varphi$$

$$dE_r = \frac{k\tau\sin\varphi dx}{\rho^2}$$

We have to substitute all variables using φ and integrate from 0 to π:

$$E_z = 2\int_0^{\frac{\pi}{2}} \frac{k\tau\sin\varphi d\varphi}{r} = \frac{2k\tau}{r} = \frac{\tau}{2\pi\varepsilon_0 r}$$

"Quiz": What was easier?

Potential of the Spherically Symmetric Field A->B

$$\left[\varphi(B)-\varphi(A)\right]=-\int_{A}^{B}\vec{E}\cdot d\vec{l} = -\int_{rA}^{rB}E(r)dr$$

We just substitute for *E(r)* and integrate:

$$\left[\varphi(B)-\varphi(A)\right]=-kQ\int_{rA}^{rB}\frac{dr}{r^{2}}=kQ(\frac{1}{r_{B}}-\frac{1}{r_{A}})$$

We see that φ decreases with *1/r* !

The Gradient I

$$grad\ f(\vec{r}) = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}](\vec{r})$$

It is a vector constructed from differentials of the function f into the directions of each coordinate axis.

It is used to estimate change of the function f if we make an elementary shift $d\vec{l}$.

The Gradient II

$$f(\vec{r}+d\vec{l}\,)=f(\vec{r}\,)+d\vec{l}\cdot grad(f(\vec{r}\,))$$

The change is the last term. It is a dot product. It is the biggest if the elementary shift $d\vec{l}$ is parallel to the grad.

In other words the grad has the direction of the biggest change of the function f!

^

The Acceleration of an *e* and *p* I

What is the acceleration of an electron and a proton in the electric field $E = 2\ 10^4\ V/m$?

$a_e = E\ q/m = 2\ 10^4\ 1.76\ 10^{11} = 3.5\ 10^{15}\ ms^{-2}$

$a_p = 2\ 10^4\ 9.58\ 10^7 = 1.92\ 10^{12}\ ms^{-2}$

$[J/Cm\ C/kg = N/kg = m/s^2]$

The Acceleration of an Electron II

What would be the speed of an electron, if accelerated from zero speed by a voltage (potential difference) of *200 V*?

$$\Delta E_k = E_k = \frac{mv^2}{2} = -(-e)V$$

$$v_e = \sqrt{\frac{2Ve}{m}} = \sqrt{400\ 1.76\ 10^{11}} = 8.39\ 10^6\, ms^{-1}$$

Thermal motion speed ~ 10^3 m/s can be neglected even in the case of protons (v_p = 1.97 10^5 m/s)!

Relativistic Effects When Accelerating an Electron

Relativistic effects start to be important when the speed reaches about 10% of the speed of light ~ $c/10 = 3\ 10^{7}\ ms^{-1}$.

What is the accelerating voltage to reach this speed?

Conservation of energy: $mv^2/2 = q\ V$

$V = mv^2/2e = 9\ 10^{14}/4\ 10^{11} = 2.5\ kV\ !$

A proton would need $V = 4.7\ MV!$

Relativistic Approach I

If we know the speeds will be relativistic we have to use the famous Einstein's formula:

$$E = mc^2 = m_0c^2 + E_K = m_0c^2 + qV$$

E is the total and E_K is the kinetic energy,

m is the relativistic and m_0 is the rest mass

$$m = \gamma m_0 \Rightarrow (\gamma - 1)m_0c^2 = qV \Rightarrow$$

$$\frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma = \frac{qV}{m_0c^2} + 1$$

Relativistic Approach II

The speed is usually expressed in multiples of the c by means of $\beta = v/c$. Since β is very close to 1 a trick has to be done not to overload the calculator.

$$\gamma = \frac{1}{\sqrt{(1-\beta^2)}} = \frac{1}{\sqrt{(1+\beta)(1-\beta)}} \cong \frac{1}{\sqrt{2(1-\beta)}}$$

So for β we have :

$$\beta = 1 - \frac{1}{2\gamma^2}$$

Example of Relativistic Approach

Electrons in the X-ray ring of the NSLS have kinetic energy $E_k = 2.8\ GeV$. What is their speed. What would be their delay in arriving to α-Centauri after light?

$E_0 = 0.51\ MeV$ for electrons. So $\gamma = 5491$ and v = 0.999 999 983 c. The delay to make $4\ ly$ is $dt = 2.1\ s$! Not bad and the particle would find the time even shorter!!

Physics

11

Electrostatics II

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Main Topics

- Electric Charge and Field in Conductors
- The Field of the Electric Dipole
- Behavior of E. D. in External Electric Field
- Examples of Some Important Fields
- An Example of Storing a Charge
- C * U = Q   :   Capacity * Voltage = Charge
- Capacitors in Series and in Parallel.
- Electric Energy Storage.
- Inserting a Conductor into a Capacitor.
- Inserting a Dielectric into a Capacitor.
- Microscopic Description of Dielectrics
- Concluding Remarks to Electrostatics.

I-5 Special Electrostatic Fields

A Charged Solid Conductor I

- Conductors contain free charge carriers of one or both polarities. Charging them means to introduce in them some excess charges of one polarity.
- A special case are metals :
  - every atom which joins metal structure, often crystallic, keeps some of its electrons in its vicinity but the valence electrons, which are bounded by the weakest forces, are shared by the whole structure and they are the free charge carriers. They can move within the crystal when electric (or other) force is acting on them.
  - It is relatively easy to add some excess free electrons to metal and also to take some out of it.

A Charged Solid Conductor II

- Adding electrons means charging the metal negatively.
- Taking some electrons out means charging it positively.
- For our purposes we can consider the ‘holes’ left after missing electrons as positive free charge carriers each with charge +1e.
- So effectively the charged metal contains excess charges either negative or positive, which are free to move.

A Charged Solid Conductor III

- Excess charges repel themselves and since they are free to move as far as to the surface, in equilibrium, they must end on a surface.
- In equilibrium there must be no forces acting on the charges, so the electric field inside is zero and also the whole solid conductor must be an equipotential region.

A Hollow Conductive Shell I

- In equilibrium again:
  - the charges must remain on the outer surface.
  - the field inside is zero and the whole body is an equipotential region.
- The above means the validity of the Gauss' law.
- To proof that let's return to the Gauss' law.

The Gauss' Law Revisited I

- Let us have a positive point charge Q and a spherical Gaussian surface of radius r centered on it. Let us suppose radial field:

$$E(r) = \frac{kQ}{r^p}$$

- The field lines are everywhere parallel to the outer normals, so the total flux is:

$$\Phi_e = E(r)A = \tfrac{1}{\varepsilon_0} Q r^{2-p}$$

- But if p≠2 the flux would depend on r !

The Gauss' Law Revisited II

- The validity of the Gauss' law $\Leftrightarrow$ $p = 2$.
- By using a concept of the solid angle it can be shown that the same is valid if the charge Q is anywhere within the volume surrounded by the spherical surface.
- By using the same concept it can be shown that the same is actually valid for any closed surface.
- It is roughly because from any point within some volume we see any closed surface confining it under the solid angle of 4π.

A Hollow Conductive Shell II

- Let first the shell be spherical. Then the charge density σ on its surface is constant.
- From symmetry, in the center the intensities from all the elementary surfaces that make the whole surface always compensate themselves and $\vec{E} = 0$
- For any other point within the sphere they compensate themselves and $\vec{E} = 0$ only if $p = 2$.
- Again, using the concept of solid angle, it can be shown, the same is valid for any closed surface.

A Hollow Conductive Shell III

- Conclusion: The existence of a zero electric field within a charged conductive shell is equivalent to the validity of the Gauss' law.
- This is the principle of:
  - experimental proof of the Gauss' law with a very high precision: $p - 2 = 2.7 \div 3.1\ 10^{-16}$.
  - of shielding and grounding (Faraday's cage).

Electric Field Near Any Conducting Surface

- Let us take a small cylinder and submerge it into the conductor so its axis is perpendicular to the surface.
- The electric field
  - within the conductor is zero
  - outside is perpendicular to the surface
- A non-zero flux is only through the outer cup $\Rightarrow$

$$E = \frac{\sigma}{\varepsilon_0}$$

- Beware the edges! σ is not generally constant!

The Electric Dipole I

- Materials can produce non-zero electric fields in their vicinity even when the total charge in them is compensated.
- But they must contain so called electric multipoles in which the centers of gravity of positive and negative charges are not in the same point.
- The fields produced are not centrosymmetric and decrease generally faster than the field of the single point charge.

The Electric Dipole II

- The simplest multipole is the electric dipole.
  - It is the combination of two charges of the same absolute value but different sign $+Q$ and $-Q$.
  - They are separated by vector $\vec{l}$, starting in $-Q$.
  - We define the dipole moment as :

$$\vec{p} = Q\vec{l}$$

The Electric Dipole III

- Electric dipoles (multipoles) are important because they are responsible for all the electrical behavior of neutral matter.
- The components of material (molecules, domains) can be polar or their dipole moment can be induced.
- Interactions of dipoles are the basis of some types of (weaker) atomic bonds.

Behavior of the Electric Dipole in External Electric Fields

- In uniform electric fields the dipoles are subjected to a torque which is trying to turn their dipole moments in the direction of the field lines
- In non-uniform electric fields the dipoles are also dragged.

Some Examples

- The field of homogeneously charged sphere
- Parallel uniformly charged planes
- Electrostatic xerox copier

I-6 Capacitance and Capacitors

Storing Charge I

- In the end of 18th century people were amazed by electricity, mainly by big discharges - sparks.
  - The entertainers had noticed that different bodies charged to the same voltage contained different amounts of “electricity” (charge in our words) and produced sparks of different impact.

Storing Charge II

- So a problem had arisen to store as big charge as possible for the maximum voltage available.
- First they needed larger and larger "containers" but after a better solution was found!

- Let's have a conductive sphere $r_i = 1\ m$.
- A quiz: If we were not limited by voltage, can we put any charge on it?

Storing Charge III

- The answer is NO!
- We are still limited by the breakdown intensity. In dry air $E_m \simeq 3\ 10^6\ V/m$.
- The maximum intensity depends on the properties of the surroundings of the conductor and the conductor itself (there would be some limit even in vacuum).
- If the maximum intensity is reached the conductor will self-discharge.
- Rough surfaces would make things even worse.

Storing Charge IV

- Using the Gauss' law: $E = 0$ within the sphere and $E = kQ/r_i^2$ close to the surface.
- From relations of the potential and the intensity $\varphi = kQ/r_i$ within and on the sphere.
- Combining these we get: $\varphi = r_i E$ for $r > r_i$
- The maximum voltage and charge for our sphere are :

$$\varphi = 3\ 10^6\ V \Leftrightarrow Q_{max} = 3.3\ 10^{-4}\ C.$$

Storing Charge V

- This voltage anyway far exceeded the limits of power sources at the time, which were, say, $10^5\ V$.
- On our sphere, such voltage would correspond to small charge : $Q = Vr_i/k = 10^5/9\ 10^9 = 1.11\ 10^{-5}\ C$.
- This could originally be improved only by increasing of the sphere r_i.
- Then someone (in Leyden, Germany) made a miracle! He inserted this sphere r_i into a little bigger one r_o, which he grounded.
- The sparks grew considerably!

Storing Charge VI

- The smaller sphere, charged with +Q, produced charge –*Q* on the inner surface of the bigger sphere and charge +*Q* in its outer surface. But when grounded the positive charge from the outer surface was repelled to the ground, so charge –*Q* remained on the outer sphere, particularly its inner surface.
- The result: potential of the charged sphere decreased, while the charge remained same!

Storing a Charge VII

- The potential from the inner sphere:

  $\varphi_i = kQ/r_i$ for $r \le r_i$; $\varphi_i = kQ/r$ for $r > r_i$

- The potential from the outer sphere:

  $\varphi_o = -kQ/r_o$ for $r \le r_o$; $\varphi_o = -kQ/r$ for $r > r_o$

- From the superposition principle:

$$\varphi(r) = \varphi_i(r) + \varphi_o(r)$$

- The potential is zero for $r \ge r_o$!

Storing a Charge VI

- So the potential on the inner sphere is here also the voltage between the spheres:

$$V_i = kQ(1/r_i - 1/r_o) = kQ(r_o - r_i)/r_i r_o$$

- Let $r_o = 1.01\ m$ and $V = 10^4\ V \Rightarrow$

  $Q = 1.12\ 10^{-3}\ C$ the charge increased $101\ x$!
- We have obtained a capacitor (condenser).
- (Q_{max} would still be $3\ 10^{-4}\ C$, however !)

The Capacitance

- The voltage between any two charged conductors is generally proportional their charge

$$Q = C\,V$$

- The constant of proportionality C is called the capacitance. It is the ability to store the charge.
- Its unit is called Farad. $1\ F = 1\ C/V$

Various Types of Capacitors

- It makes sense to produce a device meant to store charge – a capacitor.
- The capacitance of capacitors should not depend on their surroundings.
- Capacitors are used to store a charge and at the same time a potential energy.
- Most widely used are parallel plate, cylindrical and spherical capacitors.

Quiz: Two Parallel Planes

- Two large parallel planes are *d* apart. One is charged with a charge density σ, the other with $-\sigma$. Let E_b be the intensity between and E_o outside of the planes. What is true?
  - A) $E_b = 0$, $E_o = \sigma/\varepsilon_0$
  - B) $E_b = \sigma/\varepsilon_0$, $E_o = 0$
  - C) $E_b = \sigma/\varepsilon_0$, $E_o = \sigma/2\varepsilon_0$

Determination of Capacitance I

- Generally, we find the dependence of Q on V and capacitance is the coefficient of the proportionality between them.
- In the case of parallel plates of area A, distance d apart, charged to $+Q$ and $-Q$:
- Gauss' law: $E = \sigma/\varepsilon_0 = Q/\varepsilon_0 A$
- Also: $E=V/d \Rightarrow Q = \varepsilon_0 AV/d \Rightarrow C = \varepsilon_0 A/d$

Determination of Capacitance II

- The potential on one sphere in the universe :

$$V_i = kQ/r_i \Rightarrow C = r_i/k$$

- The second "electrode" of this "capacitor" is the infinity or more probably ground, which is closer. The capacitance would be strongly influenced by presence of conductors in the near neighborhood.

Determination of Capacitance III

- In the case of spherical capacitor we had :

$V_i = kQ(1/r_i - 1/r_o) = kQ(r_o - r_i)/r_i r_o$

which corresponds to the capacitance :

$$C = \frac{r_i r_o}{k(r_o - r_i)} = \frac{4\pi\varepsilon_0 r_i r_o}{(r_o - r_i)}$$

The capacitance doesn't depend on near conductors, unless they are very close.

Charging a Capacitor

- To charge a capacitor we
  - either connect the capacitor to a power source one plate to its plus the other plate to its minus pole. The first will be charged with the positive charge the other with the negative charge. The voltage of the power source will be across the capacitor, when equilibrium is reached.
  - or we ground (temporarily) one plate and charge the other as in our example.

Capacitors in Series I

- Let us have two capacitors C_1 a C_2 in series. We can replace them by a single capacitance

$$C_s = \frac{C_1 C_2}{C_1 + C_2}$$

  - If we charge one end while the other is grounded, both (all) capacitors will be charged by induction having the same charge :

$$Q = Q_1 = Q_2 = \ldots$$

Capacitors in Series II

- Connected electrodes must be at the same potential so the total voltage is the sum of voltages on both (all) capacitors :

$$V = V_1 + V_2 \Rightarrow$$

$$\frac{1}{C_s} = \frac{V}{Q} = \frac{V_1}{Q} + \frac{V_2}{Q} = \frac{1}{C_1} + \frac{1}{C_2}$$

Capacitors in Parallel

- Let us have two capacitors C_1 a C_2 in parallel. We can replace them by one capacitance C_p :

$$C_p = C_1 + C_2$$

  - Total charge is distributed between both (all) capacitors.

$$Q = Q_1 + Q_2$$

  - Voltage on both (all) capacitors is the same

$$V = V_1 = V_2 \Rightarrow$$

$$C_p = Q/V = Q_1/V + Q_2/V = C_1 + C_2$$

Absolute limit of charge

- Capacitance of a parallel plate capacitor (in vacuum) can be increased both by increasing the area of the plates and decreasing of their distance. Only the first way, however, leads to decrease of the electric field and thereby to increase the absolute limit of the the charge which can be stored!
- It would be actually better to ground the inner and charge the outer sphere of our spherical capacitor.

I-7 Electric Energy Storage. Dielectrics.

Electric Energy Storage I

- We have to do work to charge a capacitor.
- This work is stored as a potential energy and all (if neglecting the losses) can be used at later time (e.g. faster to gain power).
- If we do any changes to a charged capacitor we do or the field does work. It has to be distinguished whether the power source is connected or not during the change.

Electric Energy Storage II

- Charging a capacitor means to take a positive charge from the negative electrode and move it to the positive electrode or to take a negative charge from the positive electrode and move it to the negative electrode.
- In both cases (we can take any path) we are doing work against the field and thereby increasing its potential energy. Charge should not physically pass through the gap between the electrodes of the capacitor!

Electric Energy Storage III

- A capacitor with the capacitance C charged by the charge Q or to the voltage V has the energy:

$$U_p = Q^2/2C = CV^2/2 = QV/2$$

- The factor ½ in the formulas reveals higher complexity then one might expect. By moving a charge between the electrodes we also change Q, V so we must integrate.

Electric Energy Storage IV

- The energy density
  - Let us have a parallel plate capacitor *A,d,C,* charged to some voltage *V*:

$$E_p = \tfrac{1}{2} C V^2 = \frac{\varepsilon_0 A E^2 d^2}{2d} = Ad \, \tfrac{1}{2} \varepsilon_0 E^2$$

  - Since *Ad* is volume of the capacitor we can treat $\varepsilon_0 E^2/2$ as the density of (potential) energy
  - In uniform field each volume contains the same energy.

Electric Energy Storage V

- In non-uniform fields energy has to be integrated over volume elements where E is (approximately) constant.
- In the case of charged sphere these volumes would be concentric spherical shells ($r > r_i$) :

$$dW(r) = 4\pi r^2 dr \frac{k^2 Q^2 \varepsilon_0}{2r^4} = \frac{kQ^2 dr}{2r^2}$$

Electric Energy Storage VI

- Integrating from some $R \geq r_i$ to infinity we get :

$$W(R,\infty) = \frac{kQ^2}{2}\int_R^\infty \frac{dr}{r^2} = \frac{kQ^2}{2R}$$

- For $R = r_i$ we get the same energy as from a formula for spherical "capacitor".
- We can also see, for instance, that half of the total energy is in the interval $r_i < r < 2r_i$ or 99% of the total energy would be in the interval $r_i < r < 99r_i$

Inserting a Conductor Into a Capacitor I

- Let us insert a conductive slab of area A and thickness $\delta < d$ into the gap between the plates of a parallel plate capacitor $A, d, \varepsilon_0, \sigma$.
- The conductive slab contains enough free charge to form on its edges a charge density σ_p equal to the original σ. So the original field is exactly compensated in the slab.
- Effectively the gap changed to $d - \delta$.

A Guiz

- Inserting a conductive slab of area A and thickness $\delta < d$ into the gap between the plates of a parallel plate capacitor $A, d, \varepsilon_0, \sigma$ will increase its capacitance.
- Where should we put the slab to maximize the capacitance ?
  - A) next to one of the plates.
  - B) to the plane of symmetry.
  - C) it doesn't matter.

C: It doesn't matter !

- Let us insert the slab some distance x from the left plate. Then we effectively have a serial connection of two capacitors, both with the same A. One has the gap x and the other d-x-δ. So we have:

$$\frac{1}{C}=\frac{x}{\varepsilon_0 A}+\frac{d-x-\delta}{\varepsilon_0 A}=\frac{d-\delta}{\varepsilon_0 A}\Rightarrow$$

$$C=\frac{\varepsilon_0 A}{d-\delta}$$

Inserting a Conductor Into a Capacitor II

- The capacitance has increased.
  - In the case of disconnected power source the charge is conserved and the energy decreases – the slab would be pulled in.
  - In the case of connected power source the voltage is conserved and the energy increases – we do work to push the slab in and also the source does work to put some more charge in.

Inserting a Dielectric Into a Capacitor I

- Let us charge a capacitor, disconnect it from the power source and measure the voltage across it. When we insert a dielectric slab we shall notice that
  - The voltage has dropped by a ratio $K = V_0/V$
  - The slab was pulled in by the field
- We call K the dielectric constant or the relative permitivity (ε_r) of the dielectric.
- ε_r depends on various qualities T, f!

Inserting a Dielectric Into a Capacitor II

- What has happened : Since the inserted plate is a dielectric it contains no free charges to form a charge density on its edges, which would be sufficient to compensate the original field.
- But the field orientates electric dipoles. That effectively leads to induced surface charge densities which weaken the original field and thereby increase the capacitance.

Inserting a Dielectric Into a Capacitor III

- The field orientates electric dipoles their charges compensate everywhere except on the edges next to the capacitor plates, where some charge density $\sigma_p < \sigma$ remains.
- The field in the dielectric is then a superposition of the field generated by the original σ and the induced σ_p charge densities.
- In the case of homogeneous polarization the induced charge density $\sigma_p = P$ which is so called polarization or the density of dipole moments.

Inserting a Dielectric Into a Capacitor IV

- Inserting dielectrics is actually the most effective way to increase the capacitance. Since the electric field decreases, the absolute “breakdown” charge increases.
- Moreover for most dielectrics their breakdown intensity (or dielectric strength) is higher than that of air. They are better insulators!

Energy Density in Dielectrics

- If we define the permitivity of a material as:

$$\varepsilon = K\varepsilon_0 = \varepsilon_r\varepsilon_0$$

and use it in all formulas where ε_0 appears .

For instance the energy density can be written as $\varepsilon E^2/2$.

If dielectrics are non-linear or/and non-uniform their description is considerably more complicated!

Capacitor Partly Filled with a Dielectrics

- If we neglect the effects near the edges of the dielectrics, we can treat the system as a serial or/and parallel combination of capacitors, depending on the particular situation.

Concluding Remarks To Electrostatics

- We have illustrated most of things on very simplified examples.
- Now we know relatively deeply all the important qualitative principles of the whole electrostatics.
- This should help us to understand easier the following parts ad well as functioning of any device based on electrostatics!

The Solid Angle I

Let us have a spherical surface of radius *r*. From its center we see an element of the surface *da* under a solid angle $d\Omega$:

$$d\Omega = \frac{da}{r^2}$$

We see the whole spherical surface under :

$$\Omega = \frac{4\pi r^2}{r^2} = 4\pi$$

The Solid Angle II

If there is a point charge Q in the center the elementary flux through da is:

$$d\Theta_e = \vec{E} \cdot d\vec{a} = E\, da\, \cos\varphi = kQ \frac{da\, \cos\varphi}{r^2}$$

Since the last fraction is $d\Omega$, the total flux is:

$$\Theta_e = kQ \oint d\Omega = kQ 4\pi = \frac{Q}{\varepsilon_0}$$

Intensities near more curved surfaces are stronger!

Let's have a large and a small conductive spheres *R*, *r* connected by a long conductor and let's charge them. Charge is distributed between them to *Q*, *q* so that the system is equipotential:

$$\frac{Q}{R} = \frac{q}{r}; \frac{a}{A} = \frac{r^2}{R^2} \Rightarrow \sigma = \frac{Q}{S}\frac{r}{R}\frac{R^2}{r^2} = \Sigma\frac{R}{r}$$

Potential of Electric Dipole I

Let us have a charge $-Q$ at the origin and a $+Q$ in $d\vec{l}$. What is the potential in $\vec{r}$? We use the superposition principle and the gradient:

$$\varphi(\vec{r}) = \varphi_{-}(\vec{r}) + \varphi_{+}(\vec{r} - d\vec{l}\,) =$$

$$-\frac{kQ}{r} + \frac{kQ}{r} + (-d\vec{l}\,)\cdot grad(\frac{kQ}{r})$$

How to calculate grad(1/r)?

r is the distance from the origin :

$$r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{1}{r} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

e.g. the first components of the gradient is :

$$\frac{\partial (x^2 + y^2 + y^2)^{-\frac{1}{2}}}{\partial x} = (-\tfrac{1}{2})(x^2 + y^2 + y^2)^{-\frac{3}{2}} 2x = \frac{-x}{r^3}$$

Potential of Electric Dipole II

The first two terms cancel:

$$\varphi(\vec{r}) = \frac{kQd\vec{l} \cdot \vec{r}}{r^3} = \frac{k\vec{p} \cdot \vec{r}}{r^3}$$

The potential has axial symmetry with the dipole in the axis and axial anti-symmetry perpendicular to it. It decreases with $1/r^2$!

Electric Dipole - The Torque

Let us have a uniform field with intensity $\vec{E}$
Forces on both charges contribute simultaneously to the torque:

$$T = 2\frac{l}{2}QE\sin\varphi$$

The general relation is a cross product:

$$\vec{T} = \vec{p} \times \vec{E}$$

^

Electric Dipole - The Drag

Let us have a non-uniform field with intensity $\vec{E}$ and a dipole parallel to a field line (-Q in the origin).

$$F = -QE(0) + QE(dl) =$$

$$-QE(0) + QE(0) + Qdl\frac{dE}{dx}$$

Generally: $$\vec{F} = grad\vec{E} \cdot \vec{p}$$

The vector ≡ cross product I

$$Let \quad \vec{c} = \vec{a} \times \vec{b}$$

Definition (components)

$$c_i = \varepsilon_{ijk} a_j b_k$$

The magnitude $|\vec{c}|$

$$|\vec{c}| = |\vec{a}||\vec{b}| \sin \varphi$$

Is the surface of a parallelepiped made by $\vec{a}, \vec{b}$.

The vector ≡ cross product II

The vector c is perpendicular to the plane made by the vectors $\vec{a}$ and $\vec{b}$ and they have to form a right-turning system.

$$\vec{c} = \begin{Vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{Vmatrix}$$

ε_{ijk} = {1 (even permutation), -1 (odd), 0 (eq.)}

Charging a Capacitor

Let at some point of charging the capacitor *C* have some voltage *V(q)* which depends on the current charge *q*. To move a charge *dq* across *V(q)* we do work $dE_p = V(q)dq$. So the total work to reach the charge *Q* is:

$$E_p = \int_0^Q V(q)dq = \tfrac{1}{C}\int_0^Q q\,dq = \frac{Q^2}{2C}$$

Polarization ≡ Dipole Moment Density I

Let us take some volume *V* which is small in the macroscopic scale but large in the microscopic scale so it is representative of the whole sample:

$$\vec{P} \equiv \frac{\sum_{V} \vec{p}}{V}$$

Polarization ≡ Induced Surface Charge Density II

Let a single dipole moment $p = lq$ be confined in a prism of the volume $v = al$. A volume V of the uniformly polarized dielectric is built of the same prisms, so the polarization must be the same as in any of them:

$$P \equiv \frac{\sum p}{V} = \frac{p}{v} = \frac{lq}{al} = \frac{q}{a} = \sigma_p$$

Polarization III

The result field in the dielectric :

$$E = E_0 - E_p = E_0 - \frac{\sigma_p}{\varepsilon_0} = E_0 - \frac{P}{\varepsilon_0}$$

We can express the original charge density:

$$\sigma = \varepsilon_0 E_0 = \varepsilon_0 E + P$$

So the original field is distributed to the result field and the polarization according to the ability of the dielectric to be polarized.

Polarization IV

In linear dielectrics $\vec{P}$ is proportional to the result field $\vec{E}$. They are related by the dielectric susceptibility χ:

$$\vec{P} = \varepsilon_0 \chi \vec{E} \Rightarrow$$

$$\varepsilon_0 E_0 = \varepsilon_0 (1 + \chi) E = \varepsilon_0 K E = \varepsilon E$$

The result field E is K times weaker than the original field E_0 and can also define the permitivity of a dielectric material as ε.

Physics

12
Electrokinetics

Stationary Electric Currents

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Main Topics

- Electric Currents = Moving Charges or Changing Electric Field
- Power Sources
- The Ohm's Law
- Transfer of Charge, Energy and Power
- Resistance and Resistors, Resistors in Series and Parallel, Resistor Networks
- General Topology of Circuits
- Kirchhoff's Laws – Physical Meaning and Use
- The superposition principle
- The Use of the Loop Currents Method
- Real Power Sources
- DC Voltmeters and Ammeters – Building and Use
- Wheatstone Bridge.
- Charging Accumulators.
- The Resistivity and Conductivity – Conductors, Semiconductors and Insulators.
- The Speed of Moving Charges.
- The Ohm's Law in Differential Form.
- The Classical Theory of Conductivity.
- The Temperature Dependence of Resistivity
- The Thermocouple

II–1 Ohm’s Law

Electric Currents I

- So far we were interested in equilibrium situations.
- Before equilibrium is reached non-zero fields exist which force charges to move so currents exist.
- On purpose we often maintain potential difference on a conductor in order to keep the currents flow.
- The current at some instant is defined as:

$$I(t) = \frac{dq(t)}{dt}$$

Electric Currents II

- From the physical point of view we distinguish three types of currents:
  - conductive – e.g. movement of charged particles in solids or solutions
  - convective – movement of charges in vacuum e.g. in the CRT- tube
  - shift – connected with changes in time of the electric field e.g. depolarization of dielectrics

Electric Currents III

- Electric currents can be realized by the movement of both types of charges.
- The conventional direction of current is in the direction of the electric field so the same way as positive charge carriers would move.
- If in the particular material the charge carriers are negative as e.g. in metals they physically move in the direction opposite to the conventional current.

Electric Currents IV

- In the rest of this lecture block we shall deal with stationary currents. This is a special case of 'semi equilibrium' when all the voltages and currents in networks we shall study are stable and constant. Stationary currents can be only conductive or convective.
- Later we shall also deal with time dependent currents, which can also include shift currents.

Electric Currents V

- The unit for the current is 1 ampere abbreviated A. $1\ A = 1\ C/s$.
- Since currents can be relatively easily measured, ampere is taken as one of the 7 main units in the SI system.
- It is used as a basis to define other electrical units e.g. 1 coulomb as $1C = 1\ As$.

Power Sources I

- To maintain a constant current e.g. a constant charge flow through a conducting rod, we have to keep the restoring field constant, which is equivalent to keep a constant potential difference between both ends of the rod or to keep a constant voltage on the rod.
- To accomplish this we need a power source.

A Quiz

- Can a charged capacitor be used as a power source to reach a stationary current?
  - A) Yes
  - B) No

The Answer

- The answer is NO! Capacitors can be used as a power sources e.g. to cover temporary drop-outs but the currents they can produce are not stationary. The current, in fact, discharges the capacitor, so its voltage decreases and so does the current.

Power Sources II

- A power source :
  - is similar to a capacitor but it must contain a mechanism, which would compensate for the discharging so a constant voltage is maintained.
  - must contain non electrical agent e.g. chemical which recharges it. It for instance moves positive charges from the negative electrode to the positive across the filed, so it does work !
  - voltage is given by the equilibrium of electric and non-electric forces.

Power Sources III

- To maintain a constant current the work has to be done at a certain rate so the power source delivers power to the conducting system.
  - There the power can be changed into other forms like heat, light or mechanical work.
  - Part of the power is unfortunately always lost as unwanted heat.

Power Sources IV

- Special rechargeable power sources exist – accumulators. Their properties are very similar to those of capacitors except they are charged and discharged at (almost) constant voltage.
- So the potential energy of an accumulator charged by some charge Q at the voltage V is $U = QV$ and not $QV/2$ as would be the case of a charged capacitor.

Ohm’s Law

- Every conducting body needs a certain voltage between its ends to build sufficient electric field to reach certain current. The voltage and current are directly proportional as is described by the Ohm’s law:

$$V = RI$$

- The proportionality parameter is called the resistance. Its unit is ohm: $1\ \Omega = 1\ V/A$

Resistance and Resistors I

- To any situation when we have a certain voltage and current we can attribute some resistance.
- In ideal resistor the resistance is constant regardless the voltage or current.
- In electronics special elements – resistors are used which are designed to have properties close to the ideal resistors.
- The resistance of materials generally depends on current and voltage.

Resistance and Resistors II

- An important information on any material is its volt-ampere characteristics.
- It is measured and conveniently plotted (as current vs. voltage or voltage vs. current) dependence. It can reveal important properties of materials.
- In any point of such characteristic we can define a differential resistance as

$$dR = \Delta V/\Delta I$$

- Differential resistance is constant for an ideal resistor.

Resistance and Resistors III

- In electronics also other special elements are used such as variators, Zener diods and varistors which are designed to have special V-A characteristics . They are used for special purposes, for instance to stabilize voltage or current.

Transfer of Charge, Energy and Power I

- Let us connect a resistor to terminals of a power source with some voltage V by conductive wires the resistance of which can be neglected. This is a very simple electric circuit.
- We see that the same voltage is both on the power source as well as on the resistor. But look at the directions of field!

Transfer of Charge, Energy and Power II

- The field will try to discharge the power source through it and also around through the circuit because this means lowering the potential energy.
- But in the power source there are non-electrical forces which actually push charges against the field so that the current flows in the same (e.g. clock wise) direction in the whole circuit.
- The external forces do work in the power source and the field does work in the resistor(s).

Transfer of Charge, Energy and Power III

- Let us take some charge dq. When we move it against the field in the power source we do work Vdq which means that the field does work $-Vdq$.
- In the resistor the field does work Vdq.
- The total work when moving the charge around the circuit is zero This, of course, corresponds to the conservativnes of the electric field.
- If we derive work by time we get power: $P = VI$.
- Counting in the resistance: $P = V^2/R = RI^2$.

Transfer of Charge, Energy and Power IV

- So power $P = VI$ is delivered by the non-electric forces in the power source, it is transported to an electric appliance by electric field and there is is again changed into non-electric power (heat, light…).
- The trick is that the power source and the appliances can be far away and it is easy to transport power using the electric field.

Transfer of Charge, Energy and Power V

- In reality the resistance of connecting wires can't be neglected, especially in the case of a long-distance power transport.
- Since the loses in the wires depend on I^2 the power is transformed to very high voltages to keep low currents and thereby to decrease the loses.

Resistors in Series

- When resistors are connected in series, they have the same current passing through them.
- At the same time the total voltage on them must be a sum of individual voltages.
- So such a connection can be replaced by a resistor whose resistance is the sum of individual resistances.

$$R = R_1 + R_2 + \ldots$$

Resistors in Parallel

- When resistors are connected in parallel, there is the same voltage on each of them.
- At the same time the total current must be a sum of individual currents.
- So such a connection can be replaced by a resistor whose reciprocal resistance is the sum of individual reciprocal resistances.

$$1/R = 1/R_1 + 1/R_2 + \ldots$$

DC Circuits I

Theory & Examples

General Resistor Networks

- First we substitute resistors in the serial branches and then in the parallel.
- A triangle circuit we replace by a star using cyclic permutations of:

$$r_\alpha = r_b r_c/(r_a + r_b + r_c)$$

- This follows from cyclic permutations of:

$$r_\alpha + r_\beta = r_c(r_a + r_b)/(r_a + r_b + r_c)$$

Example I-1 (26-19)

- Connecting R_2 means increasing current I_1 as well as V_1 and power delivered by the source. Voltages $V_3 = V_4$ must drop.
- Before connecting $I_1 = 45/150 = 0.3\ A$ and $I_3 = I_4 = I_1/2 = 0.15\ A$; $P = VI_1 = 13.5\ W$; $I_2 = 0$; $V_2 = 0$.
- After connecting $I_{1a} = 45/133.3 = 0.3375\ A$; $I_{2a} = I_{3a} = I_{4a} = I_{1a}/3$; $P = VI_{1a} = 15.2\ W$ etc.

Example II-1 (26-29)

- The easiest is to substitute e.g. left triangle into a star with resistances $9.09,\ 3.6,\ 4.5\ \Omega$.
- Then we add the resistors from the right triangle and find the total resistance of the system $R_t = 12.12\ \Omega$ and the total current.
- Then we go backwards finding voltages in each point and calculating currents.

General Topology of Circuits

- Circuits are constructed of
  - Branches – wires with power sources and resistors.
  - Junctions– points in which at least three branches are connected.
  - Loops – all different possible closed trips through various branches and joints which don't cross.

Solving Circuits

- To solve a circuit completely means to find currents in all branches. Sometimes it is sufficient to deal only with some of them.
- When solving circuits it is important to find independent loops. There are geometrical methods for that and usually several possibilities.
- In practice, we have to obtain enough linearly independent equations for currents.

The Kirchhoff's Laws I

- The physical background for solving circuits are the Kirchhoff's laws. They express fundamental properties the conservation of charge and potential energy.
- In the simplest form they are valid in the approximation of stationary fields and currents but can be generalized to some time dependent fields and currents as well.

The Kirchhoff's Laws II

- The Kirchhoff's first law or junction rule states that at any junction point, the sum of all currents entering the junction must be equal the sum of all currents leaving the junction.
- It is a special case of conservation of charge which is more generally described by the equation of continuity of the charge.

The Kirchhoff’s Laws III

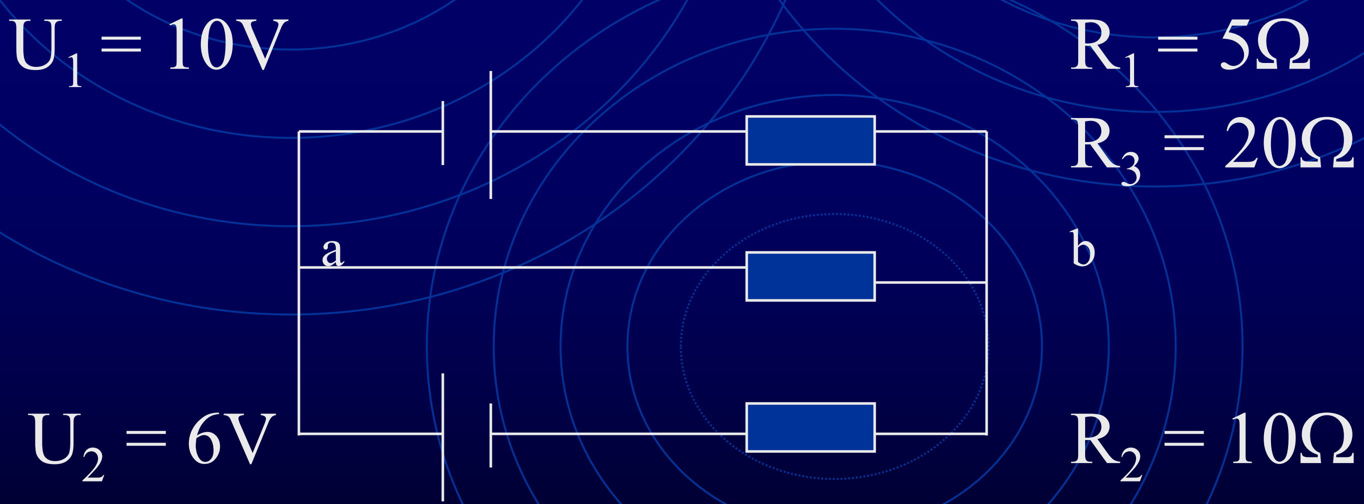
- The Kirchhoff’s second law or loop rule states that, the sum of all the changes in potential around any closed path (= loop) of a circuit must be zero.
- It is based on the conservation of potential energy or more generally on the conservativity of the electric field.

The Use of Kirchhoff's Laws I

- We have to build as many independent equations as is the number of branches
  - First we name all currents and choose their direction. If we make a mistake they will be negative in the end.
  - We write equations for all but one junctions. The last equation would be lin. dependent.
  - We write equation for every independent loop.

Example III-1

- Our circuit has 3 branches, 2 junctions and 3 loops of which two are independent.
- Since there are sources in two branches we can't use simple rules for serial or parallel connections of resistors.



Example III-2

- We name the currents and choose their directions. Here, let all leave the junction *a*, so at least one must be negative in the end.
- It is convenient to mark polarities on resistors according to the supposed direction of currents.
- The equation for the junction *a* is :

$$I_1 + I_2 + I_3 = 0.$$

Example III-3

- Equation for the junction *b* would be the same so we must proceed to loops.
- We e.g. start in the point *a* go through the branch 1 and return through the branch 3:

$$-V_1 + R_1 I_1 - R_3 I_3 = 0$$

- Similarly from *a* via 2 and back via 3:

$$V_2 + R_2 I_2 - R_3 I_3 = 0$$

Example III-4

- The "rule of the thumb" is to put all terms on one side of the equation and write the sign according to the polarity which we approach first during the path.
- Then we can get $-I_3 = I_1 + I_2$ from the first equation and substitute it the the other two:

$$V_1 = (R_1 + R_3)I_1 + R_3I_2$$

$$-V_2 = R_3I_1 + (R_2 + R_3)I_2$$

Example III-5

- Numerically we have:

$$25I_1 + 20I_2 = 10$$

$$20I_1 + 30I_2 = -6$$

- We can proceed several ways and finally get: $I_1 = 1.2\ A$, $I_2 = -1\ A$, $I_3 = -0.2\ A$
- We see that the current I_2 and I_3 run the opposite direction the we had estimated.

The Use of Kirchhoff's Laws II

- The Kirchhoff's laws are not really useful for practical purposes because they require to build and solve as many independent equations as is the number of branches. But it can be shown the it is sufficient to build and solve just as many equations as is the number of independent loops, which is always less.

Example IV-1

- Even in our simple example we had to solve a system of three equations which is the limit which can be relatively easily solved by hand.
- We can show that even for a little more complicated circuit the number of equations would be enormous and next to impossible to solve.

Example IV-2

- Now we have 6 branches, 4 junctions and 7 loops out of which 3 are independent.
- Kirhoff’s laws give us 3 independent equations for junctions and 3 for loops.
- We have a system of 6 equations for 6 currents, which is in principle enough but it would be very difficult to solve it.

The Principle of Superposition I

- The superposition principle can be applied in such a way that all sources act independently.
- We can shortcut all sources and leave only the *j-th* on and find currents I_{ij} in every branch.
- We repeat this for all sources. Then for current in i-th branch $I_i = I_{i1} + I_{i2} + I_{i3} + \ldots$

The Principle of Superposition II

- A simple illustration: Let's have a power source of *12 V*, its positive electrode is connected to the positive electrode of a second power source of *6 V*. Both negative electrodes are connected via a *3 Ω* resistor.
- The first p. source creates a current $I_1 = +4\ A$
- The second p. source creates a current $I_2 = -2\ A$
- Since the sources act together the total current is

$$I = I_1 + I_2 = +2\ A$$

Example III-6

- Let us return to our first example.
- Let’s leave the first source on and shorten the second one.
- We obtain a simple pattern of resistors which we easily solve:
- I_{11}= 6/7 A; I_{21}= -4/7 A; I_{31}= -2/7 A

Example III-7

- We repeat this for the second source:
- I_{12}= 12/35 A; I_{22}= -3/7 A; I_{32}= 3/35 A
- Totally we get:
- I_1= 1.2 A; I_2= -1 A; I_{32}= -0.2 A
- Which is the same as the previous result.
- Using superposition is handy if we want to see what happens e.g. if we double the voltage of the first source.

The Loop Currents Method

- There are several more advanced methods which use only the minimum number of equations necessary to solve the circuits.
- Probably the most elegant and easiest to understand and use is the method of loop currents.
- It is based on the idea that only currents in the independent loops exist and the other currents are their superposition.

Example III-8

- In our first example two independent loop currents exist e.g. I_α in the loop *a(1)(3)* and I_β in the loop *a(2)(3)*.
- All branch currents written as their superposition:
- $I_1 = I_\alpha$
- $I_2 = I_\beta$
- $I_3 = -I_\alpha - I_\beta$

Example III-9

- Now we write loop equations.
- $(R_1 + R_3)I_\alpha + R_3 I_\beta = V_1$
- $R_3 I_\alpha + (R_2 + R_3)\, I_\beta = -V_2$
- By inserting the numerical values and solving we get: $I_\alpha = 1.2\ A$ and $I_\beta = -1A$ which gives again the same branch currents: $I_1 = 1.2$ A, $I_2 = -1$ A, $I_3 = -0.2$ A

Example III-10

- They are, of course, the same as before but we solved only system of two equations for two currents. We skipped the step of substituting for the current I_3.
- To see the advantage even better let's revisit the fourth more complicated example.

DC Circuits II

Applications

Example IV-3

- Let I_α be the current in the DBAD, I_β in the DCBC and I_γ in the CBAC loops. Then:
- $I_1 = I_\beta - I_\alpha$
- $I_2 = I_\gamma - I_\beta$
- $I_3 = I_\alpha - I_\gamma$
- $I_4 = -I_\beta$
- $I_5 = I_\alpha$
- $I_6 = I_\gamma$

Example IV-4

- The loop equation in DBAD would be:
- $-V_1 + R_1(I_\alpha - I_\beta) - V_3 + R_3(I_\alpha - I_\gamma) + R_5 I_\alpha = 0$
- $(R_1 + R_3 + R_5)I_\alpha - R_1 I_\beta - R_3 I_\gamma = V_1 + V_3$
- Similarly from the loops DCBD and CABC:
- $-R_1 I_\alpha + (R_1 + R_2 + R_4)I_\beta - R_2 I_\gamma = V_4 - V_1 - V_2$
- $-R_3 I_\alpha - R_2 I_\beta + (R_2 + R_3 + R_6)I_\gamma = V_2 - V_3$
- It is some work but we have a system of only three equations which we can solve by hand!

Example IV-5

- Numerically we get:

$$\begin{bmatrix} 12 & -2 & -5 \\ -2 & 14 & -10 \\ -5 & -10 & 25 \end{bmatrix} \begin{bmatrix} I_\alpha \\ I_\beta \\ I_\gamma \end{bmatrix} = \begin{bmatrix} 51 \\ -16 \\ 25 \end{bmatrix}$$

- From here we get I_α, I_β, I_γ and then using them finally the branch currents I_1 …

Real Power Sources I

- Power sources have some forces of non-electric character which compensate for discharging when current is delivered.
- Real sources are not able to compensate totally. Their terminal voltage is a decreasing function of current.
- Most power source behave linearly. It means we can describe their properties by two parameters, according to a model which describes them.

Real Power Sources II

- Most common model is to substitute a real source by serial combination of an ideal power source of some voltage ϵ or EMF (electro-motoric force) and an ideal, so called, internal resistor. Then the terminal voltage can be expressed:

$$V(I) = \epsilon - R_i I$$

- If we compare this formula with behavior of a real source, we see that ϵ is the terminal voltage for zero current and R_i is the slope of the function.

Real Power Sources III

- $\in$ can be obtained only by extrapolation to zero current.
- From the equation we see that the internal resistance R_i can be considered as a measure, how close is the particular power source to an ideal one. The smaller value of R_i the closer is the plot of the function to a constant function, which would be the behavior of an ideal power source – whose terminal voltage doesn’t depend on current.

Real Power Sources IV

- The model using ϵ and R_i can be used both when charging or discharging the power source. The polarity of the potential drop on the internal resistor depends on the direction of current.
- Example: When charging a battery by a charger at $V_c = 13.2\ V$ the $I_c = 10\ A$ was reached. When discharging the same battery the terminal voltage $V_d = 9.6\ V$ and current $I_d = 20\ A$. Find the ϵ and R_i.

Real Power Sources V

- Charging:

$$\in + I_c R_i = V_c$$

- Discharging:

$$\in - I_d R_i = V_d$$

- Here:
- $\in + 10\ R_i = 13.2$
- $\in - 20\ R_i = 9.6$
- $\in = 12\ V$ and $R_i = 0.12\ \Omega$

DC Voltmeters and Ammeters I

- Measurements of voltages and currents are very important not only in physics and electronics but in whole science and technology since most of scientific and technological quantities (such as temperature, pressure …) are usually converted to electrical values.
- Electric properties can be easily transported and measured.

DC Voltmeters and Ammeters II

- In the following part we shall first deal with the principles of building simple measuring devices.
- Then we shall illustrate some typical problems which stem from non-ideality of these instruments which influences the accuracy of the measured values.

Building V-meters and A-meters I

- The heart of voltmeters or ammeters is so called galvanometer. It is a very sensitive voltmeter or ammeter. It is usually characterized by full-scale current or f-s voltage and internal resistance.
- Let us have a galvanometer of the full-scale current of $I_f = 50\ \mu A$ and internal resistance $R_g = 30\ \Omega$. Ohms law $\Rightarrow V_f = I_f R_g = 1.5\ mV$

Building V-meters and A-meters II

- If we want to measure larger currents, we have to use a shunt resistor which would bypasses the galvanometer and takes around the superfluous current.
- For instance let $I_0 = 10\ mA$. Since it is a parallel connection, at $V_f = 1.5\ mV$, there must be $I = 9.950\ mA$ passing through it, so $R = 0.1508\ \Omega$.
- Shunt resistors have small resistance, they are precise and robust.

Building V-meters and A-meters III

- If we what to measure larger voltages we have to use a resistor in series with the galvanometer. On which there would be the superfluous voltage.
- Lets for instance measure $V_0 = 10\ V$. Then at $I_f = 50\ \mu A$ there must be $V = 9.9985\ V$ on the resistor. So $R_v = 199970\ \Omega$.
- These serial resistors must be large and precise.

Using V-meters and A-meters I

- Due to their non-ideal internal resistance voltmeters and ammeters can influence their or other instruments reading by a systematic error!
- What is ideal?
- Voltmeters are connected in parallel. They should have infinite resistance not to bypass the circuit.
- Ammeter are connected in serial. They should have zero resistance so there is no voltage on them.

Using V-meters and A-meters II

- Let us measure a resistance by a direct measurement. We can use two circuits.
- In the first one the voltage is measured accurately but the internal resistance of voltmeter (if infinity) makes the reading of current larger. The measured resistance is underestimated.
- Can be accepted for very small resistances.

Using V-meters and A-meters III

- In the second scheme the current is measured accurately but the internal resistance of the ammeter (if not zero) makes the reading of voltage larger. The measured resistance is overestimated.
- Can be accepted for very large resistances.
- The internal resistances of the meters can be obtained by calibration.

Using V-meters and A-meters IV

- Normal measurements use some physical methods to get information about unknown properties of samples.
- Calibration is a special measurement done on known (standard) sample to obtain information on the method used.

Wheatstone Bridge I

- One of the most accurate methods to measure resistance is using the Wheatstone Bridge.
- It is a square circuit of resistors. One of them is unknown. The three other must be known and one of the three must be variable. There is a galvanometer in one diagonal and a power source in the other.

Wheatstone Bridge II

- During the measurement we change the value of the variable resistor till we balance the bridge, which means there is no current in the diagonal with the galvanometer. It is only possible if the potentials in the points *a* and *b* are the same:
- $I_1R_1 = I_3R_3$ and $I_1R_2 = I_3R_4$ divide them $\Rightarrow$
- $R_2/R_1 = R_4/R_3$ e.g. $\Rightarrow$ $\mathrm{R}_4 = \mathrm{R}_2\mathrm{R}_3/\mathrm{R}_1$

II–4 Microscopic View of Electric Currents

The Resistivity and Conductivity I

- Let's have an ohmic conductor i.e. the one which obeys the Ohm's law:

$$V = RI$$

- The resistance R depends both on the geometry and the physical properties of the conductors. If we have a homogeneous conductor of the length l and the cross-section A we can define the resistivity ρ and its reciprocal the conductivity γ by:

$$R = \rho \frac{l}{A} = \frac{1}{\gamma} \frac{l}{A}$$

The Resistivity and Conductivity II

- The resistivity is the ability of materials to defy the electric current. With the same geometry a stronger field is necessary if the resitivity is high to reach a certain current.
- The SI unit of resistivity is $1\ \Omega m$.
- The conductivity is the ability to conduct the electric current.
- The SI unit of conductivity is $1\ \Omega^{-1}m^{-1}$.
- A special unit siemens exists $1\ Si = \Omega^{-1}$.

Mobile Charge Carriers I

- Generally, they are charged particles or pseudo-particles which can move freely in conductors.
- They can be electrons, holes or various ions.
- The conductive properties of materials depend on how freely their charge carriers can move and this depends on deep structure properties of the particular materials.

Mobile Charge Carriers II

- E.g. in solid conductors each atom shares some of its electrons, those least strongly bounded, with the other atoms.
- In zero electric field these electrons normally move chaotically at very high speeds and undergo frequent collisions with the array of atoms of the solid. It resembles thermal movement of gas molecules ↔ electron gas.

Mobile Charge Carriers III

- In non-zero field the electrons also have some relatively very low drift speed in the opposite direction then has the field.
- The collisions are the predominant mechanism for the resistivity (of metals at normal temperatures) and they are also responsible for the power loses in conductors.

Differential Ohm's Law I

- Let us again have a conductor of the length l and the cross-section A and consider only one type of charged carriers and a uniform current, which depends on their:
  - density n i.e. number in unit volume
  - charge q
  - drift speed v_d

Differential Ohm's Law II

- Within some length Δx of the conductor there is a charge:

$$\Delta Q = n\, q \Delta x\, A$$

- The volume which passes some plane in 1 second is $A\Delta x/\Delta t = v_d A$ so the current is:

$$I = \Delta Q/\Delta t = n\, q\, v_d A = j\, A$$

- Where j is so called current density. Using Ohm's law and the definition of the conductivity:

$$I = j\, A = V/R = El\, \gamma\, A/l \Rightarrow j = \gamma E$$

Differential Ohm's Law III

$$j = \gamma E$$

- This is Ohm's law in differential form.
- It has a similar form as the integral law but it contains only microscopic and non-geometrical parameters.
- So it is a the starting point of theories which try to explain conductivity.
- Generally, it is valid in vector form: $\vec{j} = \gamma \vec{E}$

Differential Ohm’s Law IV

- Its meaning is that the magnitude of the current density is directly proportional to the field and that the charge carriers move along the field lines.
- For deeper insight it is necessary to have at least rough ideas about the magnitudes of the parameters involved in the Ohm's law.

An Example I

- Let us have a current of *10 A* running through a copper conductor with the cross-section of $3\ 10^{-6}\ m^2$.

  What is the charge density and drift velocity if every atom contributes by one free electron?
  - The atomic weight of Cu is *63.5 g/mol.*
  - The density $\rho = 8.95\ g/cm^3$.

An Example II

- 1 m^3 contains $8.95\ 10^6/63.5 = 1.4\ 10^5\ mol.$
- If each atom contributes by one free electron, this corresponds to $n = 8.48\ 10^{28}$ electrons/m^3.

$$v_d = \frac{I}{Anq} =$$

$$10/(8.48\ 10^{28}\ 1.6\ 10^{-19}\ 3\ 10^{-6}) = 2.46\ 10^{-4}\ m/s$$

The Internal Picture

- The drift speed is extremely low. It would take the electron 68 minutes to travel 1 meter! In comparison, the average speed of the chaotic movement is of the order of 10^6 *m/s*.
- So we have currents of the order of 10^{12} *A* running in random directions and so compensating themselves and relatively a very little currents caused by the field.
- It is similar as in the case of charging something a very little un-equilibrium.

A Quiz

- The drift speed of the charge carriers is of the order of 10^{-4} *m/s*.

  Why it doesn't take hours before a bulb lights when we switch on the light?

The Answer

- By switching on the light we actually connect the voltage across the wires and the bulb and thereby create the electric field which moves the charge carriers. But the electric field spreads with the speed of light $c = 3\ 10^8\ m/s$, so all the charges start to move (almost) simultaneously.

The Classical Model I

- Let's try to explain the drift speed using more elementary parameters. We suppose that during some average time between the collision τ the charge carriers are accelerated by the field. And non-elastic collision stops them.
- Using what we know from electrostatics:

$$v_d = qE\tau/m$$

The Classical Model II

- We substitute the magnitude of the drift velocity into the formula for the current density:

$$j = n\, q\, v_d = n\, q^2\, \tau\, E/m$$

- So we obtain conductivity and resistivity:

$$\gamma = \mathrm{n\, q^2}\, \tau\, /\mathrm{m}$$

$$\rho = 1/\gamma = m/nq^2\,\tau$$

The Classical Model III

- It may seem that we have just replaced one set of parameters by another.
- But here only the average time is unknown and it can be related to mean free path and the average thermal speed using well established theories similar to those studying ideal gas properties.
- This model predicts dependence of the resistivity on the temperature but not on the electric field.

Temperature Dependence of Resistivity I

- In most cases the behavior is close to linear.
- We define a change in resistivity in relation to some reference temperature t_0 (0 or 20° C):

$$\Delta \rho = \rho(t) - \rho(t_0)$$

- The relative change of resistivity is directly proportional to the change of the temperature:

$$\frac{\Delta\rho}{\rho(t_0)} = \alpha(t - t_0) = \alpha\Delta t \Rightarrow$$

$$\rho(t) = \rho(t_0)(1 + \alpha\Delta t)$$

Temperature Dependence of Resistivity II

- α [K^{-1}] is the linear temperature coefficient.
  - It is given by the temperature dependence of n and v_d.
  - It can be negative e.g. in the case of semiconductors (but exponential behavior).
- In larger temperature span we have to add a quadratic term etc.

$$\Delta\rho/\rho(t_0) = \alpha(t - t_0) = \alpha\,\Delta t + \beta\,(\Delta t)^2 + \ldots \Rightarrow$$

$$\rho(t) = \rho(t_0)(1 + \alpha\,\Delta t + \beta\,(\Delta t)^2 + \ldots)$$

The Thermocouple I

- The thermocouple is an example of a transducer, a device which transfers some physical quality (here temperature) to an electrical one.
- Unlike other temperature sensors e.g. the platinum thermometer or thermistor which use the thermal conductivity change of metals or semiconductors, the thermocouple is a power-source.

The Thermocouple II

- It is based on thermoelectric or Seebeck (Thomas 1821) effect : If we keep a difference of temperature on two ends of a conductive wire also potential difference appears between these ends.
- This voltage is proportional to the temperature difference and some a material parameter Seebeck’s coefficient.

The Thermocouple III

- Let's connect two conductors A and B in one point, which we keep at temperature t_1.
- The other ends, which are at room temperature t_0 will have voltages with respect to their contact point :

$$V_A = k_A(t_1 - t_0) \quad \text{and} \quad V_B = k_B(t_1 - t_0)$$

- A voltmeter connected between these ends shows :

$$V_{AB} = V_B - V_A = (k_B - k_A)(t_1 - t_0)$$

The Thermocouple IV

- As a thermocouple two wires with sufficiently different Seebeck's coefficient can be used.
- Usually around ten selected pairs of materials are frequently used. They are named *J, K* … and their calibration parameters are known. They differ e.g. in temperature span where they are used.
- When using one thermocouple its voltage depends on room temperature which is not a very convenient property.

The Thermocouple V

- A simple possibility to get rid of this dependence is to use a pair of thermocouples.
  - Let use make a second connection of conductors *A* and *B* and place it into known temperature t_2.
  - The we cut one of the conductors (e.g. *B*) in a place on room temperature t_0. The voltages of the points of disconnection *X* and *Y* with respect to the first common point is :

$$V_X = k_B(t_1 - t_0)$$

$$V_Y = k_A(t_1 - t_2) + k_B(t_2 - t_0)$$

The Thermocouple VI

- And the voltage between these points is :

$$V_{XY} = V_Y - V_X = k_A(t_1 - t_2) + k_B(t_2 - t_0) - k_B(t_1 - t_0)$$

so finally : $V_{XY} = (k_A - k_B)(t_1 - t_2)$

- The dependence on the room temperature has really vanished. The price is the necessity to use a bath with the reference temperature t_2. Usually some well defined phase transitions e.g. (melting of ice in water) are used. But care has to be taken e.g. for pressure dependence.

The Thermocouple VII

- Modern instruments (equipped with microprocessors) usually measure the room temperature, so they can simulate the “cold junction” (reference junction) and using only one thermocouple is sufficient.
- They can be, however, only used with the types of thermocouples for which they are preprogrammed and instructions how to precisely connect the thermocouple have to be obeyed.

Peltier's Effect

- Thermoelectric effect works also the other way. If current flows through a junction of two different materials, heat can be transferred into or from this junction.
- This is so called Peltier effect (Jean 1834).
- Peltier cells are commercially available.
  - They can be used to control conveniently temperature of some volume of interest in a temperature span of circa – 50 to 200 °C. They can both heat and cool!
  - In special cases e.g. in space ships they can even be used as power sources.

The vector or cross product I

Let ***c***=***a***.***b***

Definition (components)

$$c_i = \varepsilon_{ijk} a_j b_k$$

The magnitude |***c***|

$$|\vec{c}| = |\vec{a}|\left|\vec{b}\right| \sin\varphi$$

Is the surface of a parallelepiped made by ***a***,***b***.

The vector or cross product II

The vector c is perpendicular to the plane made by the vectors ***a*** and ***b*** and they have to form a right-turning system.

$$\vec{c} = \begin{Vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{Vmatrix}$$

ε_{ijk} = {1 (even permutation), -1 (odd), 0 (eq.)}

Physics

13

Magnetism

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Main Topics

- Introduction into Magnetism
- Permanent Magnets and Magnetic Fields
- Magnetic Induction
- Electric Currents Produce Magnetic Fields
- Forces on Electric Currents
- Forces on Moving Electric Charges
- Biot-Savart Law
- Ampere's Law
- Calculation of Some Magnetic Fields
- Magnetic Dipoles
    - The Fields they Produce
    - Their Behavior in External Magnetic Fields
- Calculation of Some Magnetic Fields
    - Solenoid
    - Toroid
    - Thick Wire with Current

Main Topics

- Applications of Lorentz Force
  - Currents are Moving Charges
  - Moving Charges in El. & Mag
  - Specific charge Measurements
  - The Story of the Electron
  - The Mass Spectroscopy
  - The Hall Effect
  - Accelerators
- Introduction to Magnetic Properties
- Magnetism on the Microscopic Scale
- Diamagnetism
- Paramagnetism
- Ferromagnetism

Introduction into Magnetism

- Magnetic and electric effects are known for many thousands years. But only in 19th century a close relation between them was found. Deeper understanding was reached only after the development of the special theory of relativity in 20th century.
- Studying of magnetic properties of materials has been up to now a field of active research.

Permanent Magnets I

- The mathematical description of magnetic fields is considerably more complicated then it is for the electric fields.
- It is worth to begin with good qualitative understanding of simple magnetic effects.
- It has been known for a long time that certain materials are capable of interacting by another long-distance force which is not electrostatic.

Permanent Magnets II

- This force had been named magnetic.
- This force can be either attractive or repulsive.
- The magnitude of this force decreases with distance.
- There had been a suspicion that electric and magnetic forces were the same thing. They are not! But they are related.

Permanent Magnets III

- The reason: magnets don’t influence charges at rest but they do influence moving charges.
- At first, the magnetic properties were attributed to some “magnetic charges”.
  - Since both attractive and repulsive forces exist there must be two kinds of these “charges”.
  - But it was found that these magnetic ”charges” can’t be separated!

Permanent Magnets IV

- If you separate a piece of any size and shape from a permanent magnet, it will always contain both “charges”. So they are called more appropriately – magnetic poles.
- Unlike poles attract and like poles repel.
- We expect that poles don’t switch without external influence and the interactions are stable.

A Simple Experiment

- The fact that unlike poles attract and like poles repel can be proved by a simple experiment using three magnets:
  - Let’s mark one pole on each of the magnets.
  - At least two of the magnets must have the mark on the same pole. We can find them using the interaction with the third magnet.
  - We readily see e.g. that marked poles repel.

Permanent Magnets V

- Around any magnet there is magnetic field which can interact with other magnets.
- In pre-physics ages it was found that the Earth is a source of a magnetic field. It is a large permanent magnet.
- A magnetic needle would always point in the North-South direction.

Permanent Magnets VI

- This is a principle of compass, used by the Chinese thousands years ago for navigation.
- A convention has been accepted:
  - the pole of a magnet pointing to the North geographic pole is called the north and the other one the south.
  - the magnetic field has the direction from the north to the south. i.e. in the direction a compass would point, which enables a simple calibration of magnets.

Permanent Magnets VII

- From this it is clear that the south magnetic pole of the Earth is near to the North geographical pole.
- A compass doesn’t point exactly to the north. It has a declination which depends on the particular location since magnetic and geographic poles dont coincide. The field is even not horizontal.
- Magnets can be imagined consisting of smaller magnets so the convention works even inside them.

Magnetic Fields I

- Similarly as in the case of electric fields, we accept an idea that magnetic interactions are mediated by magnetic fields.
- Every source of magnetic field e.g. magnet spreads (by the speed of light) around an information on its position, orientation and strength. This information can be received by another source. The results is that a force between those sources appears.

Magnetic Fields II

- As can be easily proved by a magnetic needle, magnetic fields generally change directions and therefore must be described in every point by some vector quantity. Magnetic fields are vector fields.
- Magnetic fields are usually described by the vector of the magnetic induction $\vec{B}$.

Magnetic Fields III

- The magnetic field lines are:
  - lines tangential to the magnetic induction in every point.
  - closed lines which pass through the space as well as through the magnets in the same direction as a north pole of a magnetic needle would point – from north to south.

Magnetic Fields IV

- Since magnetic monopoles don’t exist, the magnetic field lines are closed lines and outside the magnets they resemble the electric field lines of an electric dipole.
- Although it is in principle possible to study directly the forces between sources of magnetic fields, it is usual to separate problems to
  - how fields are produced
  - how they interact with other sources.

Electric Currents Produce Magnetic Fields I

- First important step to find relations between electric and magnetic fields was the discovery done by Hans Christian Oersted (1777-1851, Danish) in 1820. He found that electric currents are sources of magnetic fields.
- A long straight wire produces magnetic field whose field lines are circles centered on it.

Electric Currents Produce Magnetic Fields II

- It is interesting that these closed field lines exist as if they were produced by some invisible magnets!
- Magnetic field due to a circular loop of wire is torroidal (doughnut).
- The direction of the field lines can be found using a right-hand rule.
- Later we shall see where this rule comes from and how these and other fields look in more detail and quantitatively.

Forces on Electric Currents I

- When it was found that electric currents are sources of magnetic fields it could have been expected that magnetic fields also exert force on currents-carrying wires.
- The interaction was also proved by Oersted and a formula for a force on a wire of $d\vec{l}$ carrying the current I was found:

$$d\vec{F} = I(d\vec{l} \times \vec{B})$$ (cross product)

Forces on Electric Currents II

- For a long straight wire which can be described by the vector $\vec{l}$ carrying the current I the integral formula is valid:

$$\vec{F} = I(\vec{l} \times \vec{B})$$

- If currents produce magnetic fields and they are also affected by them it logically means that currents act on currents by magnetic forces.

Forces on Electric Currents III

- Now, we can qualitatively show that two parallel currents will attract them selves and the force will be in the straight line which connect these currents.
- This seems to be similar to a force between two point charges but now the force is the result of a double vector product as we shall see soon.

Forces on Electric Currents IV

- From the formula describing force on electric currents the units can be derived.
- The SI unit for the magnetic induction ***B*** is *1 Tesla*, abbreviated as T, $1T = 1\ N/Am$
- Some older are units still commonly used for instance *1 Gauss*: $1G = 10^{-4}\ T$

Magnetic Fields Due to Currents

Forces on Moving Electric Charges I

- Since currents are in reality moving charges it can be expected that all what is valid for interaction of magnetic fields with currents will be valid also for moving charges.
- The force $\vec{F}$ of a magnetic field $\vec{B}$ acting on a charge q moving by a velocity $\vec{v}$ is given by the Lorentz formula:

$$\vec{F} = q(\vec{v} \times \vec{B})$$

Forces on Moving Electric Charges II

- Lorentz force is in fact part of a more general formula which includes both electric and magnetic forces:

$$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})]$$

- This relation can be taken as a definition of electric and magnetic forces and can serve as a starting point to study them.

Forces on Moving Electric Charges III

- Lorentz force is a central issue in whole electromagnetism. We shall return to it by showing several examples. Moreover we shall find out that it can be used as a basis of explanation of almost all magnetic and electromagnetic effects.
- But at this point we need to know how are magnetic fields created quantitatively.

Biot-Savart Law I

- There are many analogies between electrostatic and magnetic fields and of course a question arises whether some analog of the Coulomb's law exists, which would describe how two short pieces of wires with current would affect themselves. It exists but it is too complicated to use. For this reason the generation and influence of magnetic fields are separated.

Biot-Savart Law II

- All what is necessary to find the mutual forces of two macroscopic wires of various sizes and shapes with currents is to employ the principle of superposition, which is valid in magnetic fields as well and integrate.
- It is a good exercise to try to make a few calculations then try do something better!

Magnetic Field Due to a Straight Wire I

- Let's have an infinite wire which we coincide with the *x*-axis. The current *I* flows in the +*x* direction. We are interested in magnetic induction in the point *P* [0, a].
- The main idea is to use the principle of superposition. Cut the wire into pieces of the same length *dx* and add contribution of each of them.

Magnetic Field Due to a Straight Wire II

- For a contribution from a single piece we use formula derived from the Biot-Savart law:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{x} \times \vec{r}_{xP}}{\left| r_{xP} \right|^3}$$

- Since both vectors which are multiplied lie in the *x, y* plane only the *z* component of $\vec{B}$ will be non-zero which leads to a great simplification. We see where the right hand rule comes from!

Magnetic Field Due to a Straight Wire III

- So a piece of the length dx with the coordinate x contributes:

$$dB_z = \frac{\mu_0 I}{4\pi} \frac{dx \sin \alpha}{r^2}$$

- Here r is the distance of dx and P and α is the angle between the line joining dx and P and the x-axis. We have to express all these quantities as a function of one variable e.g. the α.

Magnetic Field Due to a Straight Wire IV

- For r we get:

$$r\sin\alpha = a \;\Rightarrow\; \frac{1}{r^2} = \frac{\sin^2\alpha}{a^2}$$

and for x and dx (- is important to get negative x at angles $\alpha < \pi/2$!):

$$\frac{x}{a} = -\cot\alpha \;\Rightarrow x = -a\cot\alpha \Rightarrow dx = \frac{a\,d\alpha}{\sin^2\alpha}$$

Magnetic Field Due to a Straight Wire V

- So finally we get:

$$B_z = \frac{\mu_0 I}{4\pi}\int_0^{\pi} \frac{\sin^2\alpha \sin\alpha \, a \, d\alpha}{a^2 \sin^2\alpha} =$$

$$\frac{\mu_0 I}{4\pi a}\int_0^{\pi} \sin\alpha \, d\alpha = \frac{\mu_0 I}{2\pi a}$$

The conclusions we can derive from the symmetry we postpone for later!

Ampère's Law

- As in electrostatics also in magnetism a law exists which can considerably simplify calculations in cases of a special symmetry and can be used to clarify physical ideas in many important situations.
- It is the Ampères law which relates the line integral of $\vec{B}$ over a closed path with currents which are surrounded by the path.

Magnetic Field Due to a Straight Wire VI

- As it is the case with using the Gauss' law, we have to find a path which is tangential to $\vec{B}$ everywhere and on which the magnitude of B is constant. So it must be a special field line. Then we can move B out of the integral, which then simply gives the length of the particular integration path.

Magnetic Field Due to a Straight Wire VII

- Let us have a long straight wire with current *I*.
- We expect *B* to depend on *r* and have axial symmetry where the wire is naturally the axis.
- The field lines, as we already know are circles and therefore our integration path will be a circle with a radius *r* equal to the distance where we want to find the field. Then: $2\pi r B(r) = \mu_0 I \Rightarrow$

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

Magnetic Field Due to a Straight Wire VIII

- The vectors of the magnetic induction $\vec{B}$ are tangents to circles centered on the wire, which thereby are the field lines, and the magnitude of B decreases with the first power of the distance.
  - It is similar as in the case of the electrostatic field of an straight, infinite and uniformly charged wire but there electric field lines were radial while here magnetic are circular, thereby perpendicular in every point.

Magnetic Field in a Center of a Square Loop of Current I

- Apparently by employing the Amperes law we have obtained the same information in a considerable easier way. But, unfortunately, this works only in special cases.
- Let's calculate magnetic induction in the center of a square loop a x a of current I. We see that it is a superposition of contributions of all 4 sides of the square but to get these we have to use the formula for infinite wire with appropriate limits.

Magnetic Field in a Center of a Square Loop of Current II

- The contribution of one side is:

$$B_z = \frac{\mu_0 I}{4\pi \frac{a}{2}} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\alpha \, d\alpha = \frac{\mu_0 I}{2\pi a}\sqrt{2}$$

etc.

Force Between Two Straight Wires I

- Let us have two straight parallel wires in which currents I_1 and I_2 flow in the same direction separated by a distance d.
- First, we can find the directions and then simply deal only with the magnitudes. It is convenient to calculate a force per unit length.

$$\frac{F}{l} = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}$$

Force Between Two Straight Wires II

- This is used for the definition of *1 ampere*:

  1 ampere is a constant current which, if maintained in two straight parallel conductors of infinite length, of negligible cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to $2\ 10^{-7}\ N$ per meter of length.

Magnetic Dipoles

Magnetic Dipoles I

- In electrostatics we defined electric dipoles. We can imagine them as solid rods which hold one positive and one negative charge of the same absolute values some distance apart.
  - Although their total charge is zero they are sources of fields with special symmetry which decrease faster than fields of point sources.
  - External electric field is generally trying to orient and shift them.

Magnetic Dipoles II

- Their analogues in magnetism are either thin flat permanent magnets or loops of current.
  - These also are sources of fields with a special symmetry which decrease faster than fields from straight currents
  - In external magnetic fields they are affected similarly as electric dipoles.
- Later we shall describe magnetic behavior of matter using the properties of magnetic dipoles.

Magnetic Dipoles III

- Let us have a circular conductive loop of the radius a and a current I flowing in it. Let us describe the magnetic field at some distance b on the axis of the loop.
- We can "cut" the loop into little pieces

  $dl = ad\chi$ and vector add their contribution to the magnetic induction using the Biot-Savart law.

Magnetic Dipoles IV

- For symmetry reasons the direction of $\vec{B}$ is the same as the direction of the z-axis perpendicular to the loop and integration in this case means only to add the projections $dB_z = dB\ sin\beta$. And from the geometry:

$$\sin\beta = a/r \ \Rightarrow 1/r^2 = \sin^2\beta/a^2$$

$$r^2 = a^2 + b^2$$

- Let us perform the integration.

Magnetic Dipoles V

- Since magnetic dipoles are sources of magnetic fields they must also be affected by them.
- In uniform magnetic field they will experience a torque directing them in the direction of the field.
- We shall illustrate it using a special case of rectangular loop a x b carrying current I.

A Quiz:

- What is the total force and torque on this rectangular loop if it lies perpendicularly to the field lines of the uniform magnetic field?
- What would be the difference, if the loop was circular?

The Answer

- Force on each side is perpendicular to it and lies in the plane of the rectangle. Its particular direction depends on the directions of the magnetic field and the current. The forces acting on opposite sides will cancel, however, since the current direction in them is the opposite.
- In a circular loop a force acting on its piece *dl* will cancel with the force acting on the piece *dl'* across the diameter.

Magnetic Dipoles VI

- Form the drawing we see that forces on the sides a are trying to stretch the loop but if it is stiff enough they cancel.
- Forces on the sides b are horizontal and the upper acts into the blackboard and the lower from the blackboard. Clearly they are trying to stretch but also rotate the loop.

Magnetic Dipoles VII

- To find the contribution of each of the b sides to the torque we have to find the projection of the force $\vec{F}_b$ perpendicularly to the loop: $T/2 = F_b sin\varphi\, a/2$
- Since both forces act in the same sense:

$$T = BIabsin\varphi$$

- We can generalize this using the magnetic dipole moment $\vec{m} = Iab\vec{m}_0$:

$$\vec{T} = \vec{m} \times \vec{B}$$

A Galvanometer

- A loop with current in a uniform magnetic field whose torque would be compensated by a spring is a possible principle of measuring the current. The scale of such device would not be linear!
- So special radial but constant (radial uniform) field is used to keep the two torque forces always perpendicular to the loop.

$$T = k\varphi = NIab \sin \tfrac{\pi}{2}$$

Magnetic Field of a Solenoid I

- Solenoid is a long coil of wire consisting of many loops.
- In the case of finite solenoid the magnetic field must be calculated as a superposition of magnetic inductions generated by all loops.
- In the case of almost infinite we can neglect effects close to the ends and use the Ampere's law in a very elegant way.

Magnetic Field of a Solenoid II

- As a closed path we choose a rectangle whose two sides are parallel to the axis of the solenoid.
- From symmetry we can expect that the field lines will be also parallel to the axis direction.
- Since the closed field lines return through the whole Universe outside the solenoid we can expect they are infinitely "diluted".

Magnetic Field of a Solenoid III

- Only the part of the path along the side inside the solenoid will make non-zero contribution to the loop integral.
- If the rectangle encircles N loops with current I and its length is l then:

$$Bl = \mu_0 NI$$

- And if we introduce the density of loops

$$n = N/l \Rightarrow B = \mu_0 nI$$

Magnetic Field of a Solenoid IV

- For symmetry reason we didn't make any assumptions about how deep is our rectangle immersed in the solenoid. So the magnetic field in the long solenoid can be expected to be uniform or homogeneous.
- Many physical methods rely on uniform magnetic field, e.g. NMR and mass spectrography.
- A reasonably uniform magnetic field can be obtained if we shorten thick solenoid and cut it into halves - Helmholtz coils.

Magnetic Field of a Toroid I

- We can think of the toroid as of a solenoid bent into a circle. Since the field lines cant escape we do not have to make any assumptions about the size.
- If the toroid has a radius R to its central field line and N loops of current I, we can simply show that all the field is inside and what is the magnitude on a particular field line.

Magnetic Field of a Toroid II

- Let's us choose the central filed line as our path then the integration simplifies and:

$$B(r)\,2\pi r = \mu_0 NI \Rightarrow B(r) = \mu_0 NI/2\pi r$$

- His is also valid for any r within the toroid.
- The field:
  - is non uniform since it depends on *r*.
  - is zero outside the loops of the toroid

Magnetic Field of a Thick Wire I

- Let's have a straight wire of a diameter R in which current I flows and let us suppose that the current density j is constant.
- We use Ampere's law. We use circular paths one outside and one inside the wire.
- Outside the field is the same as if the wire was infinitely thin.
- Inside we get linear dependence on r.

Magnetic Field of a Thick Wire II

- If we take a circular path of the radius r inside the wire we get:

$$B\,2\pi r = \mu_0 I_{enc}$$

- The encircled current I_{enc} depends on the area surrounded by the path

$$I_{enc} = j\,\pi r^2 = I\pi r^2/\pi R^2 \Rightarrow$$

$$B = \mu_0 I r/2\pi R^2$$

Application of Magnetic Fields

Lorenz Force Revisited

- Let us return to the Lorentz force:

$$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})]$$

and deal with its applications.

- Let's start with the magnetic field only. First, we show that

$$\vec{F} = q(\vec{v} \times \vec{B}) \Leftrightarrow \vec{F} = I(\vec{L} \times \vec{B})$$

Currents are Moving Charges I

- Let's have a straight wire with the length L perpendicular to magnetic field and charge q, moving with speed v in it.
- Time it takes charge to pass L is: $t = L/v$
- The current is: $I = q/t = qv/L \Rightarrow q = IL/v$
- Let's substitute for q into Lorentz equation:

$$F = qvB = ILvB/v = ILB$$

Currents are Moving Charges II

- If we want to know how a certain conductor in which current flows behaves in magnetic field, we can imagine that positive charges are moving in it in the direction of the current. Usually, we don’t have to care what polarity the free charge carriers really are.
- We can illustrate it on a conductive rod on rails.

Currents are Moving Charges III

- Let's connect a power source to two rails which are in a plane perpendicular to the magnetic field. And let's lay two rods, one with positive free charge carriers and the other with negative ones.
- We see that since the charges move in the opposite directions and the force on the negative one must be multiplied by -1, both forces have the same direction and both rods would move in the same direction . This is a principle of electro motors.

Moving Charge in Magnetic Field I

- Let's shoot a charged particle q, m by speed v perpendicularly to the field lines of homogeneous magnetic field of the induction B.
- The magnitude of the force is $F = qvB$ and we can find its direction since FvB must be a right-turning system. Caution negative q changes the orientation of the force!
- Since F is perpendicular to v it will change permanently only the direction of the movement and the result is circular motion of the particle.

Moving Charge in Magnetic Field II

- The result is similar to planetary motion. The Lorentz force must act as the central or centripetal force of the circular movement:

$$mv^2/r = qvB$$

- Usually r is measured to identify particles:

$$r = \frac{v}{\frac{q}{m}} \frac{1}{B}$$

- r is proportional to the speed and indirectly proportional to the specific charge and magnetic induction.

Moving Charge in Magnetic Field III

- This is basis for identification of particles for instance in bubble chamber in particle physics.
  - We can immediately distinguish polarity.
  - If two particles are identical than the one with larger r has larger speed and energy.
  - If speed is the same, the particle with larger specific charge has smaller r.

Specific Charge Measurement I

- This principle can be used to measure specific charge of the electron.
- We get free electrons from hot electrode (cathode), then we accelerate them forcing them to path across voltage V, then let them fly perpendicularly into the magnetic field B and measure the radius of their path r.

Specific Charge Measurement II

- From: $mv^2/r = qvB \Rightarrow v = rqB/m$
- This we substitute into equation describing conservation of energy during the acceleration:
- $mv^2/2 = qV \Rightarrow q/m = 2V/(rB)^2$
- Quantities on the right can be measured. B is calculated from the current and geometry of the magnets, usually Helmholtz coils.

Specific Charge of Electron I

- Originally J. J. Thompson used different approach in 1897.
- He used a device now known as a velocity filter.
  - If magnetic field B and electric field E are applied perpendicularly and in a right direction, only particles with a particular velocity v pass the filter.

Specific Charge of Electron II

- If a particle is to pass the filter the magnetic and electric forces must compensate:

$$qE = qvB \Rightarrow v = E/B$$

- This doesn't depend neither on the mass nor on the charge of the particle.

Specific Charge of Electron III

- So what exactly did Thompson do? He:
  - used an electron gun, now known as CRT.
  - applied zero fields and marked the undeflected beam spot.
  - applied electric field E and marked the deflection y.
  - applied also magnetic field B and adjusted its magnitude so the beam was again undeflected.

Specific Charge of Electron IV

- If a particle with speed v and mass m flies perpendicularly into electric field of intensity E, it does parabolic movement and its deflection after a length L:

$$y = EqL^2/2mv^2$$

- We can substitute for $v = E/B$ and get:

$$m/q = L^2B^2/2yE$$

Mass Spectroscopy I

- The above principles are also the basis of an important analytical method mass spectroscopy. Which works as follows:
  - The analyzed sample is ionized or separated e.g. by GC and ionized.
  - Then ions are accelerated and run through a velocity filter.
  - Finally the ion beam goes perpendicularly into magnetic field and number of ions v.s. radius *r* is measured.

Mass Spectroscopy II

- The number of ions as a function of specific charge is measured and on its basis the chemical composition can be, at least in principle, reconstructed.
- Modern mass spectroscopes usually modify fields so the *r* is constant and ions fall into one aperture of a very sensitive detector.
- But the basic principle is anyway the same.

The Hall Effect I

- Let's insert a thin, long and flat plate of material into uniform magnetic field. The field lines should be perpendicular to the plane.
- When current flows along the long direction a voltage across appears.
- Its polarity depends on the polarity of free charge carriers and its magnitude caries information on their mobility.

The Hall Effect II

- The sides of the sample start to charge until a field is reached which balances the electric and magnetic forces:

$$qE = qv_d B$$

- If the short dimension is L the voltage is:

$$V_h = EL = v_d BL$$

Accelerators

- Accelerators are built to provide charged particles of high energy. Combination of electric field to accelerate and magnetic field to focus (spiral movement) or confine the particle beam in particular geometry.
  - Cyclotrons
  - Synchrotrons

Cyclotrons I

- Cyclotron is a flat evacuated container which consists of two semi cylindrical parts (Dees) with a gap between them. Both parts are connected to an oscillator which switches polarity at a certain frequency.
- Particles are accelerated when they pass through the gap in right time. The mechanism serves as an frequency selector. Only those of them with frequency of their circular motion equal to that of the oscillator will survive.

Cyclotrons II

- The radius is given by:
- $r = mv/qB \Rightarrow$
- $\omega = v/r = qB/m \Rightarrow$
- $f = \omega/2\pi = qB/2\pi m$
- f is tuned to particular particles. Their final energy depends on how many times they cross the gap. Limits: size $E_k \sim r^2$, relativity

Magnetic Properties of Materials

Introduction Into Magnetic Properties I

- Magnetic properties of materials are generally more complicated than the electric ones even on the macroscopic scale. We had conductors in which the electric field was zero and dielectrics (either polar or non-polar), in which the field was always weakened. Other behaviour is rare. More subtle differences can be revealed only by studying thermal or frequency properties.

Introduction Into Magnetic Properties II

- If a material is exposed to an external magnetic field is gets magnetized and an internal magnetic field $\vec{B}_m$ appears in. It can be described as the density of magnetic dipole moments:

$$\vec{B}_m = \frac{\sum \vec{m}}{V}$$

- The volume V is small on macroscopic but large on the atomic scale.

Introduction Into Magnetic Properties III

- The total field in the magnetized material can be then written as a superposition of the original field $\vec{B}_0$ and internal field $\vec{B}_m$:

$$\vec{B} = \vec{B}_0 + \vec{B}_m$$

- Here, we can shall deal only with linear behavior:

$$\vec{B}_m = \chi_m \vec{B}_0$$

- The parameter χ_m is the magnetic susceptibility which can now be greater or less than zero.

Introduction Into Magnetic Properties IV

- We can combine these equations:

$$\vec{B} = (1+\chi_m)\vec{B}_0 = \mu_r \vec{B}_0$$

  and define the relative permeability K_m , usually also written as μ_r.
- The absolute permeability is defined as:

$$\mu = \mu_0 \mu_r = \mu_0 K_m$$

- The internal field of a long solenoid with a core can then be written as: $B = \mu n I$.

Introduction Into Magnetic Properties V

- Three common types of magnetic behavior exist. The external field in materials can be
  - weakened ($\chi_m < 0$ or $K_m < 1$) this is called diamagnetism
  - slightly intensified, ($\chi_m > 0$ or $K_m > 1$) this is called paramagnetism
  - considerably intensified, ($\chi_m >> 0$ or $K_m >> 1$) this is called ferromagnetism.

Introduction Into Magnetic Properties VI

- If a material can be ferromagnetic is is a dominant behavior which masks other behavior (diamagnetism) that is also always present but is much weaker.
- But the dominant behavior may disappear with high temperature. Ferromagnetism changes to paramagnetisms above Courie's temperature.

Magnetism on Microscopic Scale I

- Magnetic behavior of materials is an open field of research. But the main types of behavior can be illustrated by means of relatively simple models. All must start from the microscopic picture.
- We know that if we cut a piece of any size and shape from a permanent magnet, we get again a permanent magnet with both poles.

Magnetism on Microscopic Scale II

- If we continue to cut a permanent magnet we would once get to the atomic scale. The question is: which elementary particles are responsible for magnetic behavior?
- We shall show that elementary magnetic dipole moment is proportional to the specific charge so electrons are responsible for the dominant magnetic properties.
- Experiments exist, however, which are sensitive to nucleus magnetic moment (NMR, Neutron Diff.).

Magnetism on Microscopic Scale III

- Electrons can generate magnetism in three ways:
  - As moving charges as current.
  - Due to their spin.
  - Due to their orbital rotation around a core.
- The later two mechanisms add together and the way it is done is responsible for magnetic behavior in particular material.

Magnetism on Microscopic Scale IV

- Electrons can be viewed as a tiny spinning negative charged particles. The quantum theory predicts spin angular momentum *s*:

$$s = h/4\pi = 5.27\ 10^{-35}\ Js$$

- Here $h = 6.63\ 10^{-34}\ Js$ is the Planck constant
- Since electron is charged it also has a magnetic dipole moment due to the spin:

$$1\ m_s = eh/4\pi m_e = 9.27\ 10^{-24}\ J/T$$

Magnetism on Microscopic Scale V

- $m_s = m_b$ is called Bohr magneton and it is the smallest magnetic dipole moment which can exist in Nature. So it serves as a microscopic unit for dipole moments.
- We see that magnetic dipole is quantised.
- Spin is a quantum effect not a simple classical rotation. Electron would irradiate energy and slow down and fall on the core.

Magnetism on Microscopic Scale VI

- When electrons are bound in atoms they also have orbital angular momentum. It also is a quantum effect.
- It is illustrative to look at a classical planetary model of electron, even if it is not realistic, to see where the dependence on the specific charge comes from.

Magnetism on Microscopic Scale VII

- Even in a very small but macroscopic piece of material there is enormous number of electrons, each having some spin and some angular momentum. The total internal magnetic field is a superposition of all electron dipole moments.
- The magnetic behavior generally depends on whether all the magnetic moments are compensated or if some residual magnetic moment remains.

Diamagnetism I

- Materials, in which all magnetic moments are exactly compensated are diamagnetic. Their internal induced magnetic field weakens the external magnetic field.
- We can explain this behavior on (non-realistic but sometimes useful) planetary model of one electron orbiting around an atom.

Diamagnetism II

- Due to an external magnetic field a radial force acts on the electron. It points toward or out of the center depending on the direction of the field. The force can’t change the radius but if it points toward the center it speeds the electron and if out it slows it. This leads to a change in the magnetic moment which is always opposite to the field. So the field is weakened.

Paramagnetism I

- Every electron is primarily diamagnetic but if atoms have internal rest magnetic dipole moment diamagnetism is masked by much stronger effects. If the spin and orbital moments in matter are not fully compensated, the atoms as a whole have magnetic moments and they behave like magnetic dipoles. They tend to line up with the external field and thereby reinforce it.

Paramagnetism II

- The measure of organizing of dipoles due to the external field depends on its strength and it is disturbed by temperature movement.
- For fields and temperatures of reasonable values Curie's law is valid:

$$B_m = CB/T$$

where C is a material parameter.

Ferromagnetism I

- If we think of magnetism, we usually have in mind the strongest effect ferromagnetism.
- In some materials (Fe, Ni, Co, Ga and many special alloys) a quantum effect, called exchanged coupling leads to rigid parallel organizing of atomic magnetic moments in spite of the randomizing tendency of thermal motions.

Ferromagnetism II

- Atomic magnetic moments are rigidly organized in domains which are microscopic but at the same time large on the atomic scale.
- Their typical volumes are $10^{-12} – 10^{-8}\ m^3$, yet they still contain $10^{17} – 10^{21}$ atoms.
- If the matter is not magnetized the moments of domains are random and compensated.

Ferromagnetism III

- In external magnetic field the domains whose moments were originally in the direction of the field grow and the magnetic moment of some other can collectively switch its direction to that of the field.
- This leads to macroscopic magnetization.

Ferromagnetism IV

- Ferromagnetic magnetization:
  - Is a strong effect $\mu_r \approx 1000$!
  - Depends on the external field.
  - Ends in saturation.
  - Has hysteresis and thereby it can be permanent.
  - Disappears if $T > T_C$, Curie's temperature.

Ferromagnetism V

- The internal magnetization is saturated at some point. That means it can't be further increased by increasing of the external field.
- The alignment at saturation can be of the order of 75%.
- The Curie's temperature for Fe is 1043 K.

Ferromagnetism VI

- The hysteresis is due the fact that domains can’t return at low temperatures and in reasonable times to their original random configuration. Due to this, so called memory effect, some permanent magnetization remains.
- This effect is widely used e.g. to store information on floppy and hard-drives.

Planetary model of a charge I

Let's have a charge q with speed v on orbit of the radius r and calculate its magnetic dipole moment $m_0 = IA$.

The area is simply $A = \pi r^2$.

To get the current we first have to find the period of rotation: $T = 2\pi r/v$.

Then if we realize that every T one charge of q passes, the current is: $I = q/T = qv/2\pi r$.

Planetary model of a charge II

Now the magnetic moment $m_0 = IA = rqv/2$.
On the other hand the angular momentum is:

$$b = mvr.$$

If we put this together, we finally get:

$$m_0 = b\, q/2m.$$

This can be generalized into a vector form:

$$\vec{m}_0 = \vec{b}\, q/2m$$

If the charge is an electron $q = -e$ so the vectors of the magnetic moment and orbital momentum have opposite directions.

Magnetic interaction of two currents I

Let us have two currents I_1 and I_2 flowing in two short straight pieces of wire $d\vec{l}_1(\vec{r}_1)$ and $d\vec{l}_2(\vec{r}_2)$ Then the force acting on the second piece due to the existence of the first piece is:

$$d\vec{F}_{12}(\vec{r}_2) = \frac{\mu_0 I_1 I_2 d\vec{l}_2 \times [d\vec{l}_1 \times (\vec{r}_2 - \vec{r}_1)]}{4\pi \,|\, \vec{r}_2 - \vec{r}_1 \,|^3}$$

This very general formula covers almost all the magnetism physics but would be hard to use in practice.

Magnetic interaction of two currents II

That is the reason why it is divided into the formula using the field (we already know):

$$d\vec{F}_{12}(\vec{r}_2) = I_2 d\vec{l}_2 \times d\vec{B}$$

and the formula to calculate the field, which particularly is the Biot-Savart law:

$$d\vec{B}(\vec{r}_2) = \frac{\mu_0 I_1 [d\vec{l}_1 \times (\vec{r}_2 - \vec{r}_1)]}{4\pi \mid \vec{r}_2 - \vec{r}_1 \mid^3}$$

Magnetic interaction of two currents III

If we realize that:

$$\vec{r}_{12}^{\,0} = \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|}$$

is a unit vector pointing in the direction from the first current $\vec{r}_1$ to the second one $\vec{r}_2$, we se that magnetic forces decrease also with the second power of the distance.

$$d\vec{B}(\vec{r}_2) = \frac{\mu_0 I_1 [d\vec{l}_1 \times \vec{r}_{12}^{\,0}]}{4\pi |\vec{r}_2 - \vec{r}_1|^2}$$

Magnetic interaction of two currents IV

The "scaling" constant $\mu_0 = 4\pi\, 10^{-7}\ Tm/A$ is called the permeability of vacuum or of free space. Some authors don't use it since it is not an independent parameter of the Nature. It is related to the permitivity of vacuum ε_0 and the speed of light c by:

$$\varepsilon_0 \mu_0 = \frac{1}{c^2}$$

Ampère's Law

Let us have none, one, two ore more wires with currents I_1, I_2 … then:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \sum I_i$$

All the current must be added but their polarities must be taken into account !

Circular Loop of Current I

$$dB = \frac{\mu_0 I dl}{4\pi r^2} = \frac{\mu_0 I a d\chi}{4\pi r^2}$$

$$dB_z = \frac{\mu_0 I a d\chi}{4\pi r^2} \sin\beta$$

$$B_z = \oint dB_z = \frac{\mu_0 I a \sin\beta}{4\pi r^2} \oint d\chi =$$

$$\frac{\mu_0 I a \sin\beta \, 2\pi}{4\pi r^2} = \frac{\mu_0 2\pi a^2 I}{4\pi r^3} =$$

$$\frac{\mu_0}{4\pi} \frac{2IA}{(a^2 + b^2)^{\frac{3}{2}}}$$

Circular Loop of Current II

$A = \pi a^2$ is the area of the loop and its normal has the z direction. We can define a magnetic dipole moment $\vec{m} = I\vec{A}$ and suppose that we are far away so $b >> a$. Then:

$$\vec{B}(b) = \frac{\mu_0}{4\pi}\frac{2\vec{m}}{b^3}$$

Magnetic dipole is a source of a special magnetic field which decreases with the third power of the distance.

The vector or cross product I

Let ***c***=***a***.***b***

Definition (components)

$$c_i = \varepsilon_{ijk} a_j b_k$$

The magnitude |***c***|

$$|\vec{c}| = |\vec{a}|\left|\vec{b}\right| \sin \varphi$$

Is the surface of a parallelepiped made by ***a***,***b***.

The vector or cross product II

The vector c is perpendicular to the plane made by the vectors ***a*** and ***b*** and they have to form a right-turning system.

$$\vec{c} = \begin{Vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{Vmatrix}$$

ε_{ijk} = {1 (even permutation), -1 (odd), 0 (eq.)}

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Physics

14
Electromagnetism

Relation of electric and magnetic fields

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Main Topics

- Introduction into Electro-magnetism.
- Faraday's Experiment.
- Moving Conductive Rod.
- Faraday's Law.
- Lenz's Law.
- Examples
- Transporting Energy.
- Counter Torque, EMF and Eddy Currents.
- Self Inductance
- Mutual Inductance, Transformers
- Energy of Magnetic Field, Energy Density of Magnetic Field
- An RC Circuit, An RL Circuit, An RLC Circuit - Oscilations

Faraday’s Law

Introduction into Electro-magnetism

- Many scientists in history were interested in relation between electric and magnetic fields. When it was known that electric currents produce magnetic fields and interact with them a natural question appeared: do also magnetic fields produce electric fields?
- Simple experiments somehow didn’t work.

Faraday’s Experiment I

- Michael Faraday (1791-1867) used two coils on a single toroidal core. He used a power-source to produce a current through the first coil and he connected galvanometer to the other coil. He probably was not the first one to find out that there was no current through the galvanometer, regardless on how strong the current was.

Faraday's Experiment II

- But he was the first who noticed that the galvanometer deflected strongly when the power source was switched on and it also deflected in the opposite direction when he opened the switch and disconnected the power source.
- He correctly concluded that the galvanometer reacts to the changes of the magnetic field.

Simple Demonstration I

- We can show the effect of electromagnetic induction and all its qualitative properties simply, using a permanent magnet and few loops of wire, connected to a galvanometer.
- If we move the magnet into the coil the galvanometer moves in one direction if we move it out the deflection direction changes, as well as it does if we turn the magnet.

Simple Demonstration II

- If we make the experiment more accurately, taking into account which pole of the magnet is the north we find out that the current has such a direction that the field it produces goes against the changes of the external field we do by moving the magnet.
- We can also notice that it is sufficient to tilt the magnet and keep in the same distance.

Moving Conductive Rod I

- Before we state the general law describing the effect it is useful to study one special case of a conductive rod of a length l moving perpendicularly to the field lines of a uniform magnetic field with a speed v.
- Let us expect positive free charge carriers in the rod. Since we force them to move in magnetic field, they experience Lorenz force.

Moving Conductive Rod II

- The charges are free in the rod so they will move and charge positively one end of the rod.
- The positive charge will be missing on the other end so it becomes negative and new electric field appears in the rod and the force it does on the charges is opposite to the Lorenz magnetic force.

Moving Conductive Rod III

- An equilibrium will be reached when the electric and magnetic forces are equal so the net force on the charges is zero and the charging of the rod thereby stops:

$$qvB = qE = qV/l \Rightarrow V = Bvl$$

- We see that the above is valid regardless on the polarity nor the magnitude of the free charge carriers.

The Magnetic Flux I

- We have seen that movement of a conductive rod in magnetic field leads to induction of a potential difference in direction perpendicular to the movement. We call this electro-motoric force EMF.
- This was a special case of change of a new quantity the flux of the magnetic induction or shortly the magnetic flux.

The Magnetic Flux II

- The magnetic flux is defined as

$$d\Phi_m = \vec{B} \cdot d\vec{A}$$

  It represents amount of magnetic induction $\vec{B}$ which flows perpendicularly through a small surface, characterized by its outer normal vector $d\vec{A}$.
- Please, repeat what exactly the scalar and the vector product of two vectors means!

The Gauss' Law in Magnetism

- The total magnetic flux through a closed surface is always equal to zero!
- This is equivalent to the fact that magnetic monopoles don't exist so the magnetic field is the dipole field and its field lines are always closed.
- Any field line which crosses any closed surface must cross it also in again somewhere else in opposite sense.

The Faraday's Law I

- The general version of Faraday's law of induction states that the magnitude of the induced EMF in some circuit is equal to the rate of the change of the magnetic flux through this circuit:

$$\varepsilon = -\frac{d\Phi_m}{dt}$$

- The minus sign describes the orientation of the EMF. A special law deals with that.

The Faraday's Law II

- The magnetic flux is a scalar product of two vectors, the magnetic induction $\vec{B}$ and $\vec{A}$ the normal describing the surface of the circuit. So in principle three quantities can change independently to change the magnetic flux:
  - B … this happens in transformers
  - A … e.g. in our example with the rod
  - relative direction of $\vec{A}$ and $\vec{B}$ … generators

The Lenz's Law

- The Lenz's law deals with the orientation or polarity of the induced EMF. It states:
- An induced EMF gives rise to a current whose magnetic field opposes the original change in flux.
- If the circuit is not closed and no current flows, we can imagine its direction if the circuit was closed.

Moving Conductive Rod IV

- Let's illustrate Lenz's law on our moving rod. Now we move it perpendicularly to two parallel rails.
- If we connect the rails on the left, the flux grows since the area of the circuit grows. The current must be clockwise so the field produced by it points into the plane and thereby opposes the grow in flux.

Moving Conductive Rod V

- If we connect the rails on the right, the flux decreases since the area of the circuit decreases. The current must be counterclockwise so the field produced by it points out of the plane and thereby opposes the decrease in flux.
- The current in the rod is in both cases the same and corresponds to the orientation of the EMF we have found previously.

Simple Demonstration III

- If we return to the demonstration with a permanent magnet and a galvanometer.
- From its deflection we can see what is the direction of the the currents in the case we approach the wire loop and the case we leave it. From this we can find which pole of the magnet is the north and verify it in the magnetic field of the Earth.

Rotating Conductive Rod I

- A conductive rod l long, is rotating with the angular speed ω perpendicularly to a uniform magnetic field B.What is the EMF?
- The rod is "mowing" the field lines so there is EMF. But each little piece of the rod moves with different speed. We can imagine the rod like many little batteries in series. So we just integrate their voltages.

Moving Conductive Rod VI

- A QUIZ :
- Do we have to do work on the conductive rod to move it in magnetic field?

Moving Conductive Rod VII

- The answer is:
- NO after the equilibrium is reached between electric and magnetic forces and net current doesn’t flow in the rod!
- The situation will change when we bridge the rails by a resistor. WHY ?

Inductance

Transporting Energy I

- The electromagnetic induction is a basis of generating and transporting electric energy.
- The trick is that power is delivered at power stations, transported by means of electric energy (which is relatively easy) and used elsewhere, perhaps in a very distant place.
- To show the principle lets revisit our rod.

Moving Conductive Rod VIII

- If the rails are not connected (or there are no rails), no work in done on the rod after the equilibrium voltage $\in$ is reached since there is no current.
- If we don't move the rod but there is a current I flowing through it, there will be a force pointing to the left acting on it. We have already shown that $F = BIl$.

Moving Conductive Rod IX

- If we move the rod and connect the rails by a resistor R, there will be current $I = \epsilon/R$ from Ohm's law. Since the principle of superposition is valid, there will also be the force due to the current and we have to deliver power to move the rod against this force: $P = Fv = BIlv = \epsilon I$, which is exactly the power dissipated on the resistor R.

Counter Torque I

- We can expect that the same what is valid for a rod which makes a translation movement in a magnetic field is also true for rotation movement.
- We can show this on rotating conductive rod. We have to exchange the translation qualities for the rotation ones:

$$P = Fv = T\omega$$

Counter Torque II

- First let us show that if we run current I through a rod of the length l which can rotate around one of its ends in uniform magnetic field B, torque appears.
- There is clearly a force on every dr of the rod. But to calculate the torque also r the distance from the rotation center must be taken into account, so we must integrate.

Counter Torque III

- If we rotate the rod and connect a circular rail with the center by a resistor R, there will be current $I = \in/R$. Due to the principle of superposition, there will be the torque due to the current and we have to deliver power to rotate the rod against this torque: $P = T\omega = BIl^2\omega/2 = \in I$, which is again exactly the power dissipated on the resistor.

Counter EMF I

- From the previous we know that the same conclusions are valid for linear as well as for rotating movement. So we return to our rod, linearly moving on rails for simplicity.
- Let us connect some input voltage to the rails. There will be current given by this voltage and resistance in the circuit and there will be some force due to it.

Counter EMF II

- After the rod moves also EMF appears in the circuit. It depends on the speed and it has opposite polarity that the input voltage since the current due to this EMF must, according to the Lenz’s law, oppose the initial current. We call this counter EMF.
- The result current is superposition of the original current and that due to this EMF.

Counter EMF III

- Before the rod (or any other electro motor) moves the current is the greatest $I_0 = V/R$.
- When the rod moves the current is given from the Kirchhof's law by the difference of the voltages in the circuit and resistance:

$$I = (V - \in)/R = (V - vBl)/R$$

- The current apparently depends on the speed of the rod.

Counter EMF IV

- If the rod was without any load, if would accelerate until the induced EMF equals to the input voltage. At this point the current disappears and so does the force on the rod so there is no further acceleration.
  - So the final speed v depends on the applied voltage V.
  - Now, we also understand that an over-loaded motor, when it slows too much or stops, can burn-out due to large current. Motors are constructed to work at some speed and withstand a certain current $I_w < I_0$.

Eddy Currents I

- So far we dealt with one-dimensional rods totally immersed in the uniform magnetic field.
- But if the conductor must be considered as two or three dimensional and/or it is not completely immersed in the field or the field is non- uniform a new effect, called eddy currents appears.

Eddy Currents II

- The change is that now the induced currents can flow within the conductor. They cause a forces opposing the movement so the movement is attenuated or power has to be delivered to maintain it.
- Eddy currents can be used for some purposes e.g. smooth braking of hi-tech trains or other movements.

Eddy Currents III

- But eddy currents produce heat so they are source of power loses and in most cases they have to be eliminated as much as possible by special construction of electromotor frames or transformer cores e.g. laminating.

The Self Inductance I

- We have shown that if we connect some input voltage to a free conductive rod immersed in external magnetic field an EMF appears which has the opposite polarity then the input voltage.
- But even a circuit of conducting wire without any external field will behave qualitatively the very same way.

The Self Inductance II

- If some current already flows through such a wire, the wire is actually immersed in the magnetic field produced by its own current.
- If we now try to change the current we are changing this magnetic field and thereby the magnetic flux and so an EMF is induced in a direction opposing the change.
- If we make N loops in our circuit, the effect is increased N times!

The Self Inductance III

- We can expect that the induced EMF in this general case depends on the:
  - geometry of the wire and material properties of the surrounding space
  - rate of the change of the current
- It is convenient to separate these effects and concentrate the former into one parameter called the (self) inductance L.

The Self Inductance IV

- Then we can simply write: $\varepsilon = -L\dfrac{dI}{dt}$
- We are in a similar situation as we were in electrostatics. We used capacitors to set up known electric field in a given region of space. Now we use coils or inductors to set up known magnetic field in a specified region.
- As a prototype coil we usually use a solenoid (part near its center) or a toroid.

The Self Inductance V

- Let's have a long solenoid with N loops.
- If some current I is flowing through it there will be the same flux Φ_{m1} passing through each loop.
- If there is a change in the flux, there will be EMF induced in each loop and since the loops are in series the total EMF induced in the solenoid will be N times the EMF induced in each loop.
- We use Faraday's law slightly modified for this situation and previous definition of inductance.

The Self Inductance VI

$$\varepsilon = -N\frac{d\Phi_{m1}}{dt} = -L\frac{dI}{dt}$$

- If N and L are constant we can integrate and get the inductance:

$$N\Phi_{m1} = LI \Leftrightarrow L = \frac{N\Phi_{m1}}{I}$$

- The unit for magnetic flux is 1 weber

$1\ Wb = 1\ Tm^2$

- The unit for the inductance is 1 henry

$1H = Vs/A = Tm^2/A = Wb/A$

The Self Inductance VII

- The flux through the loops of a solenoid depends on the current and the field produced by it and the geometry. In the case of a solenoid of the length l and cross section A and core material with μ_r:

$$N\Phi_{m1} = NA\frac{\mu_0 \mu_r NI}{l} \Rightarrow L = \frac{\mu A N^2}{l}$$

- In electronics compomemts having inductance inductors are needed and are produced.

The Mutual Inductance I

- In a similar way we can describe mutual influence of two inductances more accurately total flux in one as a function of current in the other.
- Let us have two coils N_i, I_i on a common core or close to each other.
- Let Φ_{21} be the flux in each loop of coil 2 due to the current in the coil 1.

The Mutual Inductance II

- Then we define the mutual inductance M_{21} as total flux in all loops in the coil 2 per the unit of current (1 ampere) in the coil 1:

$$M_{21} = N_2\Phi_{21}/I_1 \Leftrightarrow I_1M_{21} = N_2\Phi_{21}$$

- EMF in the coil 2 from the Faraday's law:

$$\epsilon_2 = - N_2 d\Phi_{21}/dt = - M_{21}\, dI_1/dt$$

- M_{21} depends on geometry of both coils.

The Mutual Inductance III

- It can be shown that the mutual inductance of both coils is the same $M_{21} = M_{12}$.
- The fact that current in one loop induces EMF in other loop or loops has practical applications. It is e.g. used to power supply pacemakers so it is not necessary to lead wires through human tissue. But the most important use is in transformers.

Energy of Magnetic Field

The Transformer I

- Transformer is a device with (usually) two or more coils sharing the same flux. The coil to which the input voltage is connected is called primary and the other(s) are secondary.
- Transformers are mostly used to convert voltages or to adjust (match) internal resistances.

The Transformer II

- Let us illustrate the principle of functioning of a transformer on a simple type with two coils with N_1 and N_2 loops. We shall further suppose that there is negligible current in the secondary coil.
- Since both lops share the same magnetic flux, in each loop of each coil the same EMF $\in_1$ is induced:

$$\in_1 = - d\Phi_{m1}/dt$$

The Transformer III

- If we connect a voltage V_1 to the primary coil, the magnetization in the core will grow until the counter EMF induced in this coil is equal to the input voltage:

$$V_1 = N_1 \in_1$$

- The voltage in the secondary coil is also proportional to its number of loops:

$$V_2 = N_2 \in_1$$

The Transformer IV

- So voltages in both coils are proportional to their number of loops:

$$V_1/N_1 = V_2/N_2$$

- It is more difficult to understand the case when the secondary coil is loaded and, of course, even much more difficult to design a good transformer with high efficiency. In big transformers it can be close to 1!

The Transformer V

- Suppose, our transformer has efficiency close to 1.
- In this case, currents are inversely proportional to the number of loops in each coil and resistances are proportional to their squares:

$$P = V_1 I_1 = V_2 N_1 I_1 / N_2 = V_2 I_2$$

$$I_1 N_1 = I_2 N_2$$

$$R_1 / N_1^2 = R_2 / N_2^2$$

Energy of Magnetic Field I

- An inductance opposes changes in current. That means it is necessary to do work to reach a certain current in a coil. This work is transferred into the potential energy of magnetic field and the field starts to return it at the moment we want to decrease the current.
- If a current I flows through a coil and we want to increase it, we have to deliver power proportional to the rate of the change we want to reach.

Energy of Magnetic Field II

- In other words we have to do work at a certain rate to move charges against the field of the induced EMF:

$$P = I\in = ILdI/dt \Rightarrow$$

$$dW = Pdt = LIdI$$

- To find the work to reach current I, we integrate:

$$W = LI^2/2$$

Energy Density of Magnetic Field I

- Similarly as in the case of a charged capacitor the energy here is distributed in the field, now of course magnetic field.
- If the field is uniform, as in a solenoid, it is easy to find the density of energy:
- We already know formulas for L and B:

$$L = \mu_0 N^2 A/l$$

$$B = \mu_0 NI/l \Rightarrow I = Bl/\mu_0 N$$

Energy Density of Magnetic Field II

$$U = \frac{\mu_0 N^2 A}{l}\left(\frac{Bl}{\mu_0 N}\right)^2 = \frac{B^2}{2\mu_0} Al$$

- Al is the inside volume of the solenoid, where we expect (most of) the field, $u_m = \frac{B^2}{2\mu_0}$ can be attributed to the energy density of the magnetic field.
- This definition is valid in generally in every point of even non-uniform magnetic field.

RC, RL, LC and RLC Circuits

- Often not only static but also kinetic processes are important. So we have to find out how quantities depend on time when charging or discharging a capacitor or a coil.
- We shall se that circuits with LC will show a new effect – un-dumped or dumped oscillations.

RC Circuits I

- Let's have a capacitor C charged to a voltage V_{c0} and at time $t = 0$ we start to discharge it by a resistor R.
- At any instant the capacitor can be considered as a power source and the Kirchhoff's loop (or Ohm's) law is valid:

$$I(t) = V_c(t)/R$$

- This leads to differential equation.

RC Circuits II

- All quantities Q, V and I decrease exponentially with the time-constant $\tau = RC$
- Now, let's connect the same resistor and capacitor serially to a power supply V_0. At any instant we find from the second Kirchhoff's law:

$$I(t)R + V_c(t) = V_0$$

a little more difficult differential equation.

RC Circuits III

- Now Q and V exponentially saturate while I exponentially decreases as in the previous case. The change of all quantities can be again described using the time-constant $\tau = RC$.

RL Circuits I

- A similar situation will be if we replace the capacitor in the previous circuit by a coil *L*.
- When the current grows the sign of the induced EMF on the coil will be the same as on the resistor and we can again use the second Kirchhof's law:

$$RI(t) + LdI/dt = V_0$$

- This is again a similar differential equation.

RL Circuits II

- The coil refuses immediate growth of the current.
- I starts from zero and exponentially saturates.
- The EMF on the coil (V_L) starts from its maximal value, equal to V_0, and exponentially decreases. When the current becomes constant, the EMF on the coil disappears.

LC Circuits I

- Qualitatively new situation appears when we connect a charged capacitor C to a inductance L.
- It can be expected that now the energy will change from the electric form to the magnetic form and back. We obtain un-dumped periodic movement.

LC Circuits II

- This circuit is called an *LC* oscillator and it produces, so called, electromagnetic oscillations.
- We can use the second Kirchhoffs law:

$$L\, dI/dt - V_c = 0$$

- This is again a differential equation but of higher order.

LC Circuits III

- What happens qualitatively:
- In the beginning the capacitor is charged and it tries to discharge through the coil. However EMF equal to the voltage on the capacitor builds on the coil to prevent quick growth of the current. The current is zero in the beginning. But its time derivative must be non-zero, so current slowly grows.

LC Circuits IV

- The capacitor discharges which causes decreases the current growth and thereby also the EMF in the coil.
- At the point, the capacitor is discharged, the voltage on it and thereby the rate of the growth of the current as well as the voltage on the coil are zero. But the current is now in its maximum and the coil prevents it to drop instantly.

LC Circuits V

- The EMF on the coil will now grow in the opposite direction to oppose the decrease of the current. But anyway the EMF as well as the decrease rate of the current grows. The capacitor will now be charged it the opposite polarity.
- At the moment the capacitor is fully charged the current is zero and everything repeats again.

LC Circuits VI

- Using formulas for electric and magnetic energy we can find:

$$U_e = \frac{Q^2}{2C} = \frac{Q_0{}^2}{2C}\cos^2(\omega t + \varphi)$$

$$U_m = \frac{LI^2}{2} = \frac{L\omega^2 Q_0{}^2}{2}\sin^2(\omega t + \varphi)$$

- Energy changes as expected from electric to magnetic and the total is constant.

LRC Circuits I

- If we add a resistor the oscillations will be dumped. The energy will be lost by thermal loses on the resistor.
- The level of dumping depends on the resistance.

The vector or cross product I

Let $\vec{c} = \vec{a} \times \vec{b}$

Definition (components)

$$c_i = \varepsilon_{ijk} a_j b_k$$

The magnitude of the vector $\vec{c}$

$$|\vec{c}| = |\vec{a}| \left| \vec{b} \right| \sin \varphi$$

Is the surface of a parallelepiped made by $\vec{a}, \vec{b}$.

The vector or cross product II

The vector $\vec{c}$ is perpendicular to the plane made by the vectors $\vec{a}$ and $\vec{b}$ and $\vec{a}, \vec{b}, \vec{c}$ must form a right-turning system.

$$\vec{c} = \begin{Vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{Vmatrix}$$

ε_{ijk} = {1 (even permutation), -1 (odd), 0 (eq.)}

The scalar or dot product

Let $c = \vec{a} \cdot \vec{b}$

Definition I. (components)

$$c = \sum_{i=1}^{3} a_i b_i$$

Definition II. (projection)

$$c = |\vec{a}| \left| \vec{b} \right| \cos \varphi$$

Can you proof their equivalence?

Gauss' Law in Magnetism

The exact definition:

$$\oint d\Phi_m = \oint \vec{B} \bullet d\vec{A} = 0$$

Rotating Conductive Rod - EMF

At first we have to deal with the directions. If the field lines come out of the plane and the rod rotates in positive direction the center of rotation will be negative. *dV* in *dr*:

$$dV = Bv(r)dr$$

And total EMF:

$$V = \int_0^l Bv(r)dr = B\omega\int_0^l rdr = \frac{B\omega l^2}{2}$$

Rotating Conductive Rod - Torque

Torque on a piece *dr* which is in a distance *r* from the center of rotation of a conductive rod *l* with a current *I* in magnetic field *B* is:

$$dT = rdF = BIrdr$$

The total torque is:

$$T = \int_0^l BIrdr = \frac{BIl^2}{2}$$

RC Circuit I

We use definition of the current $I = -dQ/dt$ and relation of the charge and voltage on a capacitor $V_c = Q(t)/C$:

$$I(t) = \frac{V_c(t)}{R} \Rightarrow \frac{dQ}{dt} = -\frac{Q(t)}{RC}$$

The minus sign reflects the fact that the capacitor is being discharged. This first order homogeneous differential equation can be solved by separating the variables.

RC Circuit II

$$\frac{dQ}{Q} = -\frac{dt}{\tau}$$

Where we define a time-constant $\tau = RC$.

We can integrate both sides of the equation:

$$\ln(Q) = \frac{-t}{\tau} + k \Longrightarrow Q(t) = Q_0 \exp(\frac{-t}{\tau})$$

The integration constant can be found from the boundary conditions $Q_0 = CV_{c0}$:

RC Circuit III

$$Q(t) = CV_{c0}\exp(\frac{-t}{\tau})$$

By dividing this by *C* and then by *R* we get the time dependence of the voltage on the capacitor and the current in the circuit.:

$$V_c(t) = V_{c0}\exp(\frac{-t}{\tau})$$

$$I(t) = \frac{V_{c0}}{R}\exp(\frac{-t}{\tau})$$

RC Circuit IV

We substitute for the current $I = +dQ/dt$ and the voltage and reorganize a little:

$$R\frac{dQ}{dt} + \frac{Q(t)}{C} = V_0$$

We get a similar equation for the charge on the capacitor but now its the right side is not zero. We can solve it by solving first a homogeneous equation and then adding one particular solution e.g. final $Q_k = CV_0$.

RC Circuit V

Since we have already solved the homogeneous equation in the previous case, we can write:

$$Q(t) = Q_0 \exp\left(\frac{-t}{\tau}\right) + CV_0$$

The integration constant we again get from the initial condition $Q(0) = 0 \Rightarrow Q_0 = -CV_0$.

RC Circuit VI

$$Q(t) = CV_0[1 - \exp(\frac{-t}{\tau})]$$

By dividing this by *C* we get the time dependence of the voltage on the capacitor:

$$V_c(t) = V_0[1 - \exp(\frac{-t}{\tau})]$$

RC Circuit VII

To get the current we have to calculate from its definition as the time derivative of the charge:

$$I(t) = \frac{dQ}{dt} = \frac{V_0}{R}\exp(\frac{-t}{\tau})$$

RL Circuit I

$$RI(t) + L\frac{dI}{dt} = V_0$$

First we solve a homogeneous equation (with zero in the right) and add a particular solution $I_m = V_0/R$ (maximal current):

$$RI(t) + L\frac{dI}{dt} = 0$$

RL Circuit II

This can be solved by separation of the variables. Using the previous, defining the time-constant $\tau = L/R$ and adding the particular solution, we get:

$$I(t) = I_0 \exp(\frac{-t}{\tau}) + I_m$$

We apply the starting conditions $I(0) = 0 \Rightarrow I_0 = -I_m$ and we get:

RL Circuit III

$$I(t) = I_m[1 - \exp(\frac{-t}{\tau})]$$

The voltage on the coil we get from

V = LdI/dt :

$$V_L(t) = V_0 \exp(\frac{-t}{\tau})$$

LC Circuit I

We use definition of the current $I = -dQ/dt$ and relation of the charge and voltage on a capacitor $V_c = Q(t)/C$:

$$\frac{d^2Q}{dt^2} + \frac{Q(t)}{LC} = 0$$

We take into account that the capacitor is discharged by positive current. This is homogeneous differential equation of the second order. We can guess the solution.

LC Circuit II

$$Q(t) = Q_0 \cos(\omega t + \varphi)$$

Now we get parameters by finding the second derivative of the *Q(t)* and substituting it into the equation:

$$-\omega^2 Q(t) + \frac{1}{LC} Q(t) = 0 \;\Rightarrow\; \omega = \sqrt{\frac{1}{LC}}$$

The solition are un-dumped harmonic oscillations.

Physics

15
Optics

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Main Topics

- Maxwell equations and E. M. Waves
- Introduction into Optics.
- Margins of Geometrical Optics.
- Fundamentals of Geometrical Optics.
- Ideal Optical System.
- Fermat's Principle.
- Reflection Optics.

Main Topics

- Refraction, Dispersion and Refraction Optics.
- Thin Lenses. Types and Properties.
- Combination of Lenses.
- Basic Optical Instruments
  - Human Eye
  - Magnifying Glass
  - Telescope
  - Microscope

Maxwell's Equations I

- All the important physics in electromagnetism can be expressed in four Maxwell's Equations, the Lorentz force and conservation of charge with interesting consequences.
  - When both electric and magnetic fields are static equations split into two independent pairs one for the electrostatic and the second for the magnetostatic field.
  - When the fields are changing in time they are bound and there is one electromagnetic field.

Maxwell's Equations II

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt}$$

$$\oint \vec{B} \cdot d\vec{s} = 0$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_e}{dt}$$

Maxwell's Equations II

- The first equation is the Gauss' law from which it follows::
  - Charges – the sources of electric fields exist.
  - If they are present field-lines start in positive and end in negative charges. In the ground or infinity both types of field-lines can start or end.
  - The field of a point source behaves as $1/r^2$.

Maxwell's Equations III

- The second equation is the Faraday's law of electromagnetic induction :
  - If magnetic field which changes in time is not present, electric field is conservative and it has a scalar potential.
  - Electric field can be also generated by time changes of magnetic field. If this is the case it is not conservative and field-lines are closed curves.

Maxwell's Equations IV

- The third equation is the Gauss' law of magnetism :
  - Point magnetic charges do not exist. The sources of magnetic field are only dipoles and higher multipoles.
  - The field of a current element behaves as $1/r^2$.

Maxwell's Equations V

- The fourth equation is the generalized Ampere's law :
  - Magnetic field is generated by currents or time changes of the electric field. The latter shift current or displacement current was discovered by J.C. Maxwell and it was a surprise.
  - Magnetic field-lines are closed curves.

Maxwell's Equations VI

- Many interesting properties of electromagnetic field can be derived from the Maxwell equations:
  - It is sufficient that only one of the fields is not static. Then also the second one can't be static and we are dealing with one electromagnetic field.
  - Electromagnetic field exists in the form of electromagnetic waves which carry energy or information and spread with the speed of light.

General properties of EMW

- The solution of Maxwell's equations with no charges and currents present leads to general wave equations.
- In vacuum EMA waves spread with the speed of light $c = 3.10^8\ m/s$
- If we use vector $\vec{c}$ that includes the direction, then vectors $\vec{c}$, $\vec{E}$, $\vec{B}$ are perpendicular and form right-handed system in every point.
- They obey the superposition principle.

Plane EMW

- We will deal with a special, yet important solution the plane waves. If they move in the direction of +x they can be described as :

$$E = E_y = E_0 sin(kx - \omega t)$$

$$B = B_z = B_0 sin(kx - \omega t)$$

  - E a B are in phase
  - vectors $\vec{c}, \vec{E}, \vec{B}$ for right-handed system
  - wave number : $k = 2\pi / \lambda$
  - angle frequency : $\omega = 2\pi f$
  - speed of the wave : $c = f\lambda = \omega / k$

The Spectrum of EMW

- It shows up that effects of seemingly different character are in fact the same EMW with 'just' different frequency and wavelength.
  - Radio waves $\lambda > 0.1$ m
  - Microwaves $10^{-1} > \lambda > 10^{-3}$ m
  - Infrared $10^{-3} > \lambda > 7\ 10^{-7}$ m
  - Visible $7\ 10^{-7} > \lambda > 4\ 10^{-7}$ m
  - Ultraviolet $4\ 10^{-7} > \lambda > 6\ 10^{-10}$ m
  - X - rays $10^{-8} > \lambda > 10^{-12}$ m
  - Gamma rays $10^{-10} > \lambda > 10^{-14}$ m

Dualism of EMW (light)

- In many experiments electromagnetic waves behave like typical waves. In other group of effect they behave like particles. This particle behavior increases with the frequency.
- EMW spread in photons with energy given by Planck's law

$$E = hf = \frac{hc}{\lambda}$$

where f is the frequency of the appropriate wave.

Introduction into geometrical Optics

Originally: Properties and Use of Light.

Now: Far More General.

Introduction into Optics I

- Since the beginning of humankind people have tried to find an answer to a simple question: What is light?
- The first important discoveries were done some three thousand years ago and recently our knowledge almost doubles every year. Yet the deep insights change slowly and the question immutably remains.

Introduction into Optics II

- For a long time it was believed that light is a flow of some microscopic particles. So called, corpuscular theory, based on this idea had been supported e.g. by Isaac Newton (1642-1727) who managed to complete the physical knowledge in several fields e.g. mechanics and gravitation. In spite of his great authority, experiments revealed clearly wave properties of light.

Introduction into Optics III

- They were ingeniously summarized by James Clerk Maxwell (1831-1879). So now we know that visible light are in fact electromagnetic waves with wavelengths of *400 – 700 nm*.
- Surprisingly the 'particle – wave problem' remains unsolved since other experiments exist, which support the particle idea.

Introduction into Optics IV

- Energy of light (generally EMW) is transferred and also absorption and emission are realized by some minimal quanta – photons.
  - They are particles whose properties depend surprisingly on the parameters of the wave:
  - speed c (they can never slow down or stop)
  - energy $U = E = hf$ ($h = 6.63\ 10^{-34} Js$ Planck's const.)
  - lin. momentum $p = E/c = h/\lambda$
  - mass $m = E/c^2 = h/\lambda c$
  - Photons are bosons, so (unlike to the case of fermions) there is no limit on number of them in the same state - laser.

De Broglie wavelength I

- So it may seem not surprising that motion of light through a lens, a hole or a set of slits is governed by wave characteristics.
- It has been confirmed that any particle can be attributed a wavelength according to the famous De Broglie's relation: $\lambda = h/p$ and has therefore also wave properties. They are detectable, however, only for very small p.

De Broglie wavelength II

- Running man (100kg, 10 m/s) $\lambda \simeq 10^{-37}$ m
- Running bug (1 g, 1 cm/s) $\lambda \simeq 10^{-29}$ m
- Running electron (m_e, 10^6 m/s) $\lambda \simeq 10^{-10}$ m
- There is no way to detect the first two wavelengths but the third is comparable with atomic distances in molecules and crystals. This is the basis of electron diffractometry.

Introduction into Optics V

- It was found that this dualism of waves and particles is an intrinsic property of the microscopic world.
- The acceptance of the idea that microscopic entities can be 'at the same time' particles and waves is a basis on which the quantum theory, is built. It is the best, yet not easy to understand, description of the microscopic world, we recently have.

Introduction into Optics VI

- Due to this dualism also the scope of optics widened. It deals with not only the behavior and use of visible light but generally all electromagnetic and other waves but also for instance with focusing particles such as electrons or neutrons.

Margins of Geometrical Optics I

- Although, optics is an extremely wide and complex scientific field, for many practical and industrial purposes its 1st approximation which is the geometrical optics can be used. The effects it deals with can be treated by pure geometry. It inherits some properties of waves, such as:
  - straight propagation,
  - independence
  - reciprocity
- Geometrical optics stops to be a good theory if wave or particle properties start to matter.

Margins of Geometrical Optics II

- Typically wave properties start to matter when the size of optical elements is comparable to the wavelength. This is the case in radio- and microwave techniques but also limits the resolution of optical instruments.
- Particle properties are detectable for EMW of high energies but in some cases also for visible light.

Margins of Geometrical Optics III

- Geometrical optics can be used when the wavelength can be considered (close to) zero, speed infinite and the energy of the electromagnetic waves is small (or materials are used where e.g. fotoeffect is negligible).
- These conditions are usually met when dealing with visible light of low intensities.

Fundamentals of G. Optics I

- First important assumption is that light travels in the form of rays. Those are lines drawn in space, which correspond to the flow of radiant energy.
  - In isotropic and homogeneous materials rays are straight lines perpendicular to the wave-fronts of the waves.
  - Rays can be treated by pure geometry.

Fundamentals of G. Optics II

- Rays can relatively easily be traced through an optical system and wave-fronts and other qualities of imaging can be reconstructed.
- Rays follow a principle of reciprocity, if a ray can pass through an optical system in one direction, it can pass also in the opposite one. This is one result of the Fermat's principle.

Fermat's Principle I

- Fermat's principle is a convenient basis for describing the very simple but also very complicated optical phenomena. It states:

  A light ray if going from point S to point P must traverse an optical path length which is stationary with respect of variations of that path.

Fermat's Principle II

- It is a heritage of wave properties which says that wave being a ray must be (almost) in-phase with the near neighboring waves.
- Often, the meaning can be interpreted in much simpler form: from all the possible waves that can travel between two points, the ray is the one, which makes its path in the (extreme) shortest time.

An Ideal Optical System I

- By an optical system we are trying to focus all rays emanating from some point S in the object space into some point P in the image space.
- If this is reached the optical system is stigmatic for these two points.
- By ideal optical system would every 3-dim region in one space be stigmatically imaged in the other region.
- The regions are interchangeable due to reciprocity.

An Ideal Optical System II

- Properties of a real optical system should be as close as possible to that of the ideal one.
- Moreover the rays in the system should be easily traceable and due to simple parametrization an simple equation should be available which would relate the positions of the object and the image.
- Optical systems are based on the effects of reflection and refraction.

Reflection I

- Let’s use the Fermat’s principle to find the law of reflection at a top of a flat surface:
- Point S is a source of many rays which spread out radially. Since the observation point P is in the same space, the ray which comes first from S to P will be the shortest one. We can find it using a trick when we reflect the point S behind the mirror.

Reflection II

- From simple geometry it follows that the angle of incidence is equal the angle of reflection. By convention in optics we measure these angles from the normal to the reflecting surface.
  - This is valid for any element of the surface.
  - If a surface of a reasonable size is smooth the reflection is specular and from *P* we can see the image of *S*, if not it is diffuse (paper, Moon)

Reflection Optics I

- Using reflection is one possibility to build optical elements, in this case various kinds of mirrors, to produce image of an object. The image can be either real, if the rays really path through it or virtual if eye, only sees the rays coming from the direction of the image.
- R. O. is important for X-rays and neutrons.

Reflection Optics II

- Every optical element has a principal axis, which is roughly the axis of its symmetry.
- If an ideal mirror is stroked by rays coming parallel with the principal axis the rays either focus in the focal point – in the case of concave mirrors or they seem to come from a virtual focal point behind the mirror, if the mirror is convex.
- Optical properties of ideal mirror are described by one parameter only, the focal length *f*, the distance of the focal point from the mirrors center.

Reflection Optics III

- The surface of an ideal mirror should be parabolic and recently, it is in principle possible to make a parabolic mirrors.
- In most applications much cheaper spherical mirrors are used but they suffer from spherical aberration and can be successfully used only for paraxial rays – those very close to the principal axis.

Reflection Optics IV

- If a spherical mirror has curvature r the focal length f in paraxial region is:

$$f = \pm r/2$$

- + for concave mirrors
- – for convex mirrors
- The treatment of convex mirrors is similar but their focal length is negative.

Reflection Optics IV

- The distance of the object d_o, the image d_i and the focal length f obey the mirror equation:

$$1/d_o + 1/d_i = 1/f$$

which can be derived from similar triangles.

- By convention all these quantities are considered positive if they are in front of the mirror.
- The properties described in this equation are used for construction of an image to an object.

Reflection Optics V

- We can also define the lateral magnification

$$m = h_i/h_0 = - d_i/d_o$$

- Recently, special optical systems are being widely developed for instance for X-rays, neutrons or fiber optics, which use total reflection which appears at very low angles of incidence on simple or multi-layer surfaces.

Basic Optical Elements and Instruments

Refraction I

- Another important basic optical effect is refraction appearing when rays pass from one material to another. Transparent materials may differ in optical density.
- The more dense material the lower is the speed of light in it. Optical density is characterized by the absolute refraction index: $n = c/v$
  - c is the speed of light in vacuum
  - v speed in the particular material.
- Frequency of the waves passing through the interface remains constant.

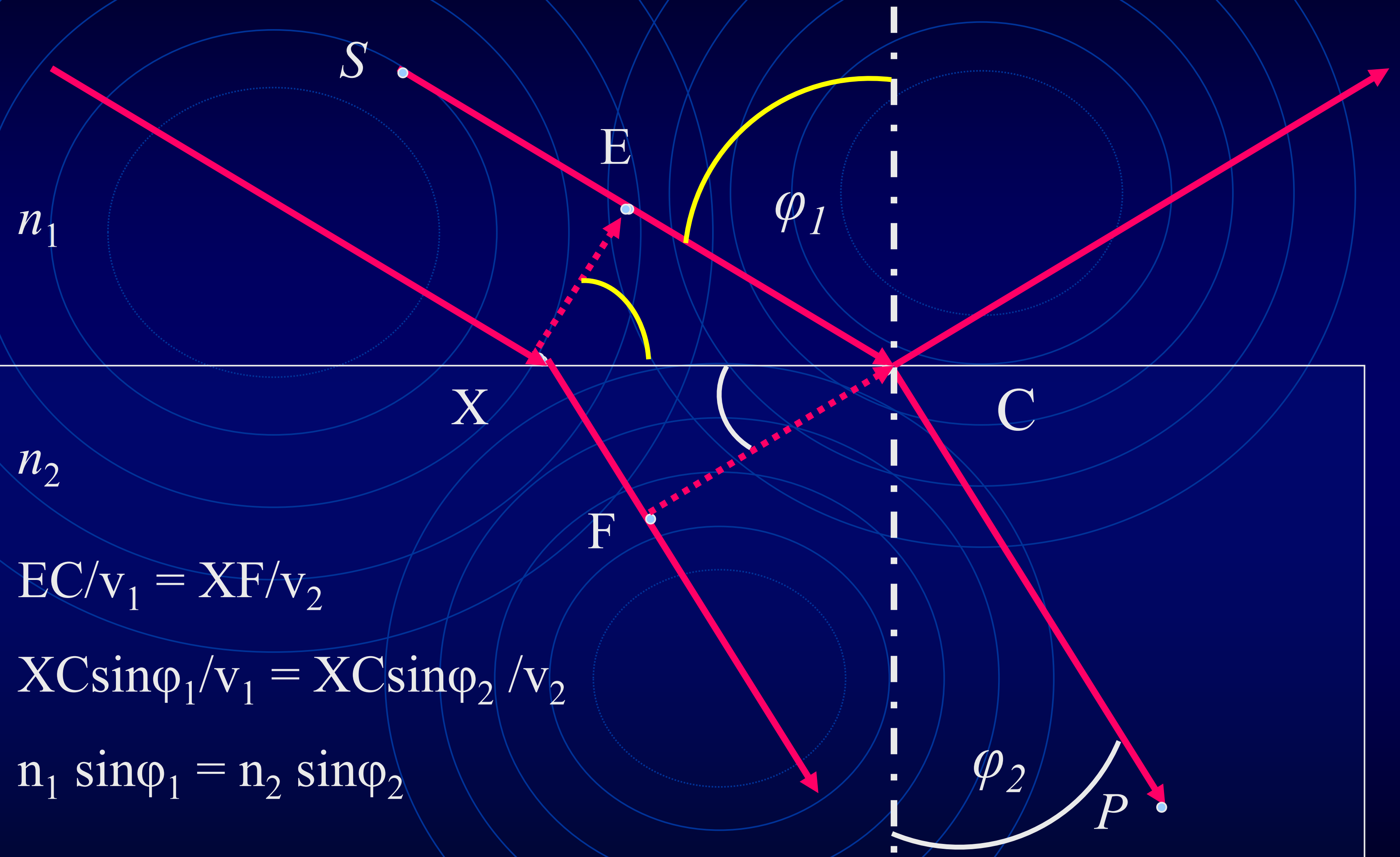
Refraction II

- We can again use the Fermat's principle to find the law of refraction.
- To find which ray makes it first from *S* to *P* is a similar problem as if we want to safe a drowning person in the shortest time, taking into account that we run much faster than swim.

Refraction III

- We use the more general definition that the correct ray is the stationary one. In other words, if we take some neighboring ray its time of flight will be (roughly) the same.
- Let the point:
  - S be in a space where the light travels with the speed $v_1 = c/n_1$ and
  - P in the space where the speed is $v_2 = c/n_2$.

Refraction IV



$EC/v_1 = XF/v_2$

$XC\sin\varphi_1/v_1 = XC\sin\varphi_2\,/v_2$

$n_1 \sin\varphi_1 = n_2 \sin\varphi_2$

Refraction V

- Now, let the *SCP* be the correct ray for and the *SXP* some neighboring ray. Should the time of flight be the same: $EC/v_1 = XF/v_2$
- We use : $EC = XC\sin\varphi_1$ and $XF = XC\sin\varphi_2$ substitute for v_1 and v_2 and get the
- Snell's law:

$$n_1 \sin \varphi_1 = n_2 \sin \varphi_2$$

Refraction VI

- We see that the higher is the optical density or the slower is the speed of light the smaller is the refraction angle.
- If the angle of incidence from the less dense material is 90° the refracted angle is given:
- $sin\varphi_c = n_1/n_2$ the maximum refracted angle or the critical angle.

Refraction VII

- If the beam would try to pass from the optically dense material under an incident angle higher than the critical angle it would not get through the boundary but rather be totally reflected.
- The effect of total (internal) reflection is <u>used</u> for instance in fiber optics.

Dispersion I

- Transparent materials have an important property that the speed of light and thereby their refraction index <u>depends</u> on the wavelength of the applied light.
- The higher energy (lower λ) the stronger interaction and thereby higher optical density and higher deflection from the original direction.
- This means that light of every wavelength or color is refracted under a (little) different <u>angle</u>.

Dispersion II

- The effect of dispersion complicates design of optical systems and has to be compensated by using more lenses of different materials.
- On the other hand it gives us the possibility to decompose the visible light and near IR and UV regions into different wavelengths.
- That is important for instance for studies of properties of matter by spectroscopic methods. The matter can be very far away in the universe!

Refraction Optics I

- The effect of refraction is used to build optical components and systems.
- If we have a point S in the medium n_1 and the point P in the medium $n_2 > n_1$ we may use the Fermat's principle to find the shape of the boundary between the media so the points are conjugated or the optical system is stigmatic for them.

Refraction Optics II

- If we compare a time of flight of some refracted ray with the one directly connecting both points we find a relation:

$$l_1 n_1 + l_2 n_2 = s_1 n_1 + s_2 n_2$$

- We readily understand from here, why the optically denser media must be convex.
- The corresponding surface is of the fourth order, so called, Cartesian ovoid.

Refraction Optics III

- If we move one of the points *S* or *P* into infinity the surface becomes second order, either elliptical or hyperbolical.
- This can be in principle used to construct lenses - optical components from some material, which allow that the object as well as the image are in the same media.

Refraction Optics IV

- Ideal lenses are for instance double hyperbolic or planar-hyperbolic.
- Although, recently they can be, in principle, machined, for the same reasons, as in the case of mirrors aspherical surfaces are approximated by cheaper spherical ones.
- But they can be successfully used only in the paraxial region.

Refraction Optics V

- Spherical surface can be shown to be stigmatic for points on the optical axis in the paraxial approximation.
- Let the the ray come from the point O in matter n_1 under an angle α and hit the spherical surface in the point P, which is seen from the curvature center C under an angle β and deflects to the point I in material n_2, where it arrives under an angle γ.
- φ_1 and φ_2 will be the incident and refracted angles.

Refraction Optics VI

- From triangle PIC : $\beta = \gamma + \varphi_2$; OPC : $\varphi_1 = \alpha + \beta$
- In paraxial approximation the angles are small, so we can write: $n_1\varphi_1 = n_2\varphi_2$

  $\alpha = h/d_0$; $\beta = h/R$; $\gamma = h/d_i$; where h is the height of the point P from the optical axis.
- We can show that the angle dependence vanishes:

$$n_1\alpha + n_2\gamma = (n_2 - n_1)\beta \Rightarrow$$

$$\frac{n_1}{d_0} + \frac{n_2}{d_i} = \frac{(n_2 - n_1)}{R}$$

Refraction Optics VII

- It is important to obey the following convention:
- If C in on the same side as the light comes from, it is negative.
- If O in on the same side as the light comes from, it is positive.
- If I in on the same side as the light comes from, it is negative.
- We can see that the reciprocity principle is valid!

Thin Lenses I

- Very important lenses are those which can be considered as thin.
  - All their properties can be characterized by a single parameter the focal length f.
  - It is the distance from the optical center to the focal points F.
  - There is one focal point in front and one behind the lens, both equally distant from the center.

Thin Lenses II

- The, so called, lensmaker's equation can be derived which relates the focal distance of a thin lens with the radii of its spherical surfaces $$P = \frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2})$$
- Sign conventions must be obeyed.
- Note that the focal length is the same on both sides even if the radii are different.

Thin Lenses III

- It is possible to make converging lenses with positive focal length when the positive radius of curvature is smaller or diverging lenses with negative focal length when the negative radius of curvature is smaller.
- Optometrist and ophthalmologist use the power $P = 1/f$ to specify lenses. Its unit is diopter (D), $1D = 1m^{-1}$.

Thin Lenses IV

- To find an image of some point, we can again use two of three special rays.
- A ray passing in any direction through optical center is not deflected.
- A ray arriving in parallel with the optical axis will pass through the image focus if f is positive or appear to leave it, if f is negative.

Thin Lenses V

- A ray passing through the object focus if *f* is positive or heading towards it, if *f* is negative, will continue in parallel with the optical axis on the other side of the lens.
- If the imaging is stigmatic (sharp) all other rays leaving the object point must appear in the image point as well. But they can't be used to find it.

Thin Lenses VI

- The lens equation which relates the distances of the object and image with the focal distance can be easily derived:

$$1/d_o + 1/d_i = 1/f$$

- and lateral magnification is defined as the ratio of the image height to the object height

$$m = h_o/h_i = - d_i/d_o$$

Thin Lenses VII

- To comprehend functioning of almost any optical instrument it is necessary to fully understand the importance of the focal planes of the lenses.
- For converging lens a bunch of parallel rays coming under some angle with the optical axis will pass through a point in the image focal plane, which is on the other side of the lens.

Thin Lenses VII

- We can locate this point using the ray passing the optical center and the one passing through the object focus.
- Using the lens equation we can verify that an object producing an image in the focal plane ($d_i = f$) must be in infinity.

Thin Lenses VIII

- For diverging lens all beams heading towards a point in the object focal plane, which is now behind the lens, will run as a bunch of parallel rays after the lens.
- We can find their direction using the ray passing the optical center and the one coming in parallel with the optical axis.

Thin Lenses IX

- We can again verify this using the lens equation. If the object is in the object focal plane ($d_o = f$, both negative now) the image must be in infinity.
- We can produce parallel bunches of rays by both types of thin lenses if the object is in the object focal plane. For the diverging lens the object distance is, however, negative!

Combination of Lenses

- We start from the lens closest to the object.
- We display the object by this lens only.
- The image of produced by the first lens will be the object for the second lens.
- Then we display the new object by the second lens only. And so on.
- The sign convention must be strictly obeyed since now object distance may be negative!

The Human Eye I

- Most of the focusing (refraction) is done by the cornea (n = *1.376*). The lens does just the 'fine tuning'.
- The quality of focusing and the depth of focus depends on the iris. The smaller the aperture the better.
- Normal eye has the near point at *25 cm* and the far point in infinity.

The Human Eye II

- In the case of nearsightedness (myopia) the far point is not infinity. This has to be corrected by a diverging lens.
- In the case of farsightedness (hyperopia or presbyopia – developed by age) the eye can't focus on near objects. This has to be corrected by a converging lens.

The Human Eye III

- The eye is relaxed if it watches the far point so eyepieces usually produce parallel rays.
- Some other optical instruments produce a virtual image in the conventional length equal to the standard near point at 25 cm.

Magnifying Glass

- Magnifying glass is used:
  - either the object is in the focal plane and we watch it by relaxed eye.
  - or the lens is close to the eye (Sherlock Holmes) and a virtual image is produced in the conventional distance.
- Magnification is the angle magnification – we see objects as big as is the angle of their image on the retina.

Telescopes I

- Astronomical refractive telescopes have two lenses an objective (with longer *f*) and an eyepiece, which share the same focal plane.
- The eyepiece can be either a converging lens or a diverging one, then the shared focal plane is behind the eyepiece.
- The angle magnification in both cases is minus the ratio of the focal lengths.

Telescopes II

- Important are reflecting telescopes:
  - Large mirrors are easier to produce and support
  - Mirrors don’t suffer from color aberration.
- But we have to realize that although reflection is not influenced by dispersion, it is still a complicated process and reflectivity of any material isn’t ideal .

Compound Microscope

- The principle of a microscope can be shown also using two lenses. The objective (now with very short *f*) produces a real image. It is watched by the eyepiece, which usually produces the imaginary image in the conventional distance.
- Good microscopes are complicated since it is important to compensate aberrations.

Lens Equation

$$\frac{h_i}{h_o} = \frac{d_i - f}{f} \wedge \frac{h_i}{h_o} = \frac{d_i}{d_o} \Rightarrow$$

$$\frac{d_i - f}{d_i f} = \frac{1}{d_o}$$

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Maxwell's Equations I

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$

$$\oint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \frac{d\Phi_e}{dt}$$

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Physics

16
Modern Physics

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Main Topics

- Particle properties of waves
  - Black body radiation – Planck law
  - Photoelectric effect
  - Compton effect
- Wave properties of particles
  - DeBroglie's waves
  - Electron diffraction
- The first models of atoms
- X-rays
- Lasers

Blackbody radiation I

- By experience we know that we are able to feel existence of a warm body close to us. This is because heat is also transferred by electromagnetic waves - radiation
- When the temperature is lower than circa 700° C the radiation is mainly in infrared range. At higher temperatures the visible part becomes more pronounced. So spectral properties of the radiation clearly depend on temperature and it makes sense to find this dependence.
- We should understand the meaning of this energy transfer by radiation: The very existence of life on our Earth depends on transfer of energy from the Sun.

Blackbody radiation II

- When studying heat radiation of a body it has to be separated from the radiation reflected from other sources. The body the radiation of which is purely heat is so called ideal black body.
- Beside the ability to emit energy every body has also ability to absorb it.
- Gustav Robert Kirchhoff has shown that these abilities are proportional, hence good absorber must also be a good emitter.

Blackbody radiation III

- In the year 1879 Josef Stefan had discovered a law that was later (1884) theoretically justified by Ludwig Boltzman :

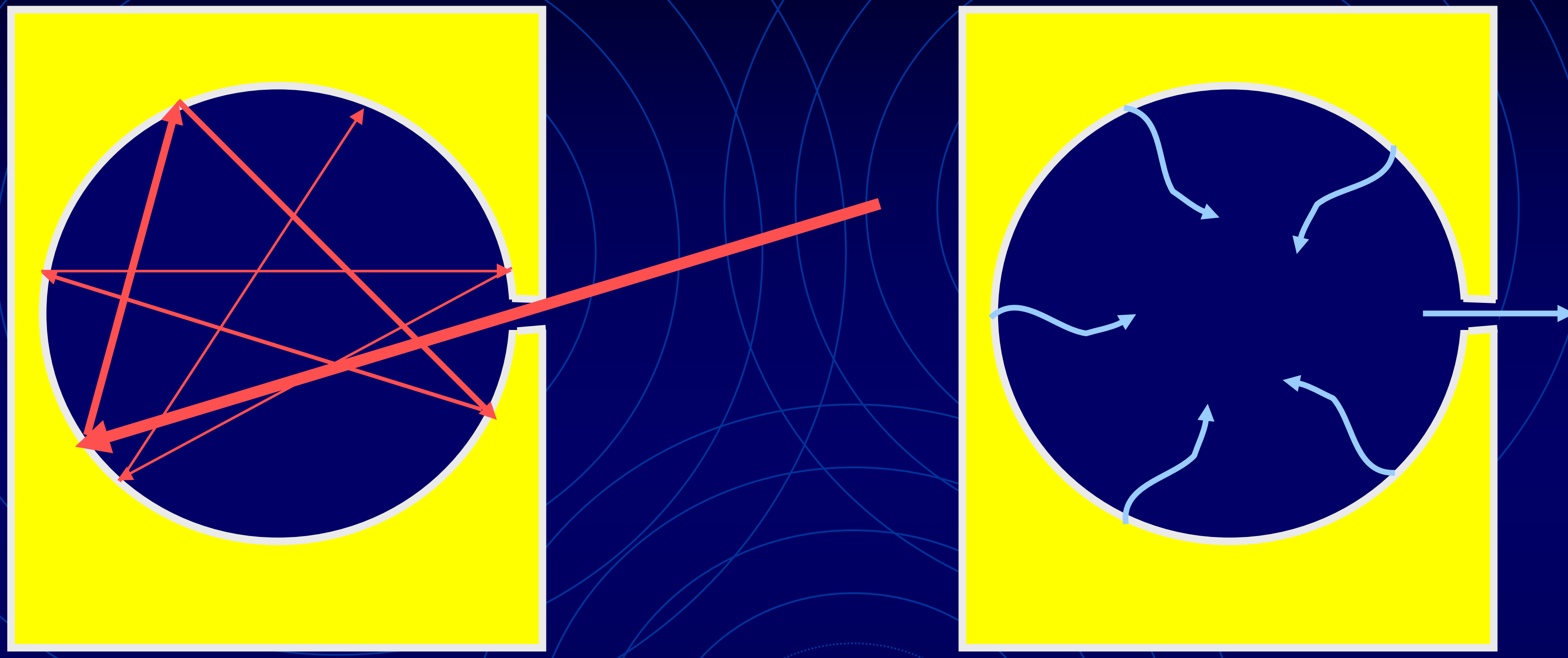
  From the surface S of material with the emittivity ε and absolute temperature T power P is emitted :

$$P = \frac{\Delta Q}{\Delta t} = \varepsilon \sigma S T^4$$

  the constant $\sigma = 5.67033\ 10^{-8}\ Wm^{-2}K^{-4}$

- So clearly we can influence cooling of a body by emitivity of its surface. To study properties of the emitter the body has to be as black as possible - the blackbody. Such a system was discovered in the 19th century.

Blackbody radiation IV



Irradiation coming from outside is totally absorbed in the cavity. (Like an eye.)

The spectrum of emitted radiation depends only on temperature of the body with the cavity.

Black body radiation V

- When the measurements were well established problems arrived with the explanation of spectral behavior as a function of temperature..
- In the year 1896 partly successful was W. Wien, who found empiric law for the behavior of maxima of spectral dependencies :

$$\lambda_m T = 2.9010^{-3}\ mK$$

λ_m is the wavelength of the maximum. This formula is the basis for remote measurements of temperature till now.

Black body radiation VI

- In the end of 19th century a theory of Rayleigh-Jeans was developped. It described well the long-wave region of the spectrum. There was, however, no theory that would describe the behavior of the whole spectrum.
- A breakthrough was the originally empiric formula, introduced by Max Planck (1885-1947) :

$$I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

$k = 1.38\ 10^{-23}\ J/K$ is Boltzman's constant and

$h = 6.626\ 10^{-34}\ J\ s = 4.1356692\ 10^{-5}\ eV\ s$ the Planck's constant

Black body radiation VII

- Planck's law is a breakthrough not only that it explains the whole spectra of heat radiation but it expects that the irradiating system consists of small oscillators the energies of these can't reach any value but are discrete - quantized :

$$E = nhf \qquad n = 1, 2, 3$$

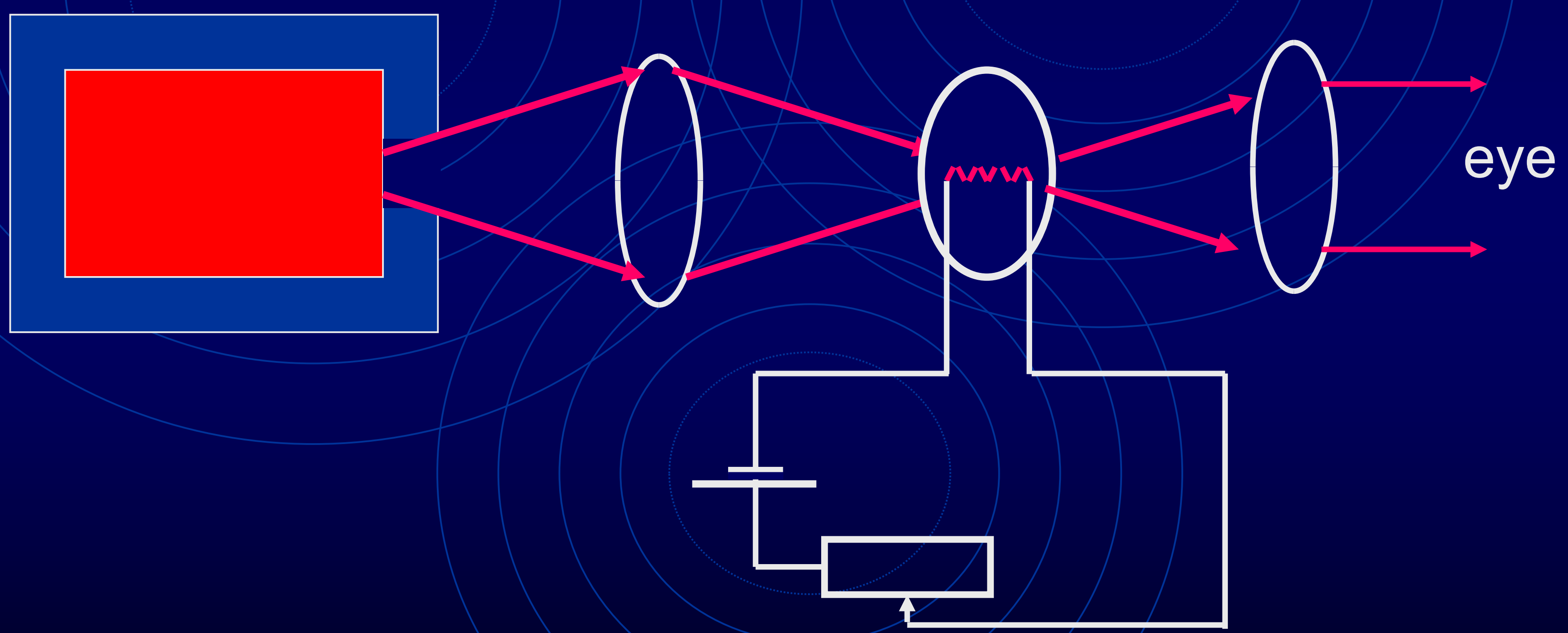
- Max Planck considered quantized energies as a trick to explain the measured data mathematically. It was Albert Einstein in 1905 who discovered the depth of the idea that energy as well as other quantities in the micro-world are quantized.

Black body radiation VIII

- Black body radiation and its interpretation by Planck's law is one of the important effects that pushed physicists for a new description of micro-world the – quantum theory.
- Beside this it can be used for remote temperature measurements from high temp furnaces to the temperatures of stars or the cosmic background radiation.

Black body radiation IX

Pyrometer with disappearing thread – temperature measurements



Photoelectric effect I

- As the name hints the photoelectric effect means emitting of electrons by matter after its irradiation by electromagnetic waves (VIS, UV).
- Let's place a test electrode near the main illuminated electrode. Almost immediately an equilibrium voltage U builds between these electrodes. This voltage corresponds to the maximum kinetic energy E_{kmax} which the electrons can have under the current conditions :

$$E_{kin} \leq eU \Rightarrow E_{k\max} = eU$$

Photoelectric effect II

- It can be shown that the E_{kmax} doesn't depend on the intensity of the electromagnetic waves it is a linear function of their frequency. Moreover electrons are emitted only when the frequency is higher than some threshold frequency. This corresponds to some minimal work W_o, necessary to release the electrons and it is a material parameter called the work function :

$$E_{k\max} = hf - W_o$$

- This again supports the idea of the existence of quanta of radiation.

Photoelectric effect III

- The wave conception of the EMW is in contradiction with the speed of the effect: If the power was distributed homogeneously in the cross section of the beam, the time needed to accumulate energy necessary to release an electron would be much longer the experiment really shows.
- The probability of a collision of very little photon and very little atom is indeed small but it is multiplied by extremely high number of photons in the beam and number of irradiated atoms.

Photoelectric effect IV

- Many every-day effects are connected with the existence of photons and dependence of their energy on their color. E.g. the use of a red light in the dark-room, of green leaves of photosynthesizing plants.
- Measurements of the energy distribution of emitted electrons is the basis of several important surface methods e.g. nanoESCA.

Compton's effect I

- It the year 1923 A. Compton found out that the wavelength of X-ray beam scattered by matter is longer than that of the primary beam and moreover it strongly depends on the scattering angle.
- It shows up that the effect is caused by inelastic collisions of electrons and photons which beside energy must also have a linear momentum.
- An example:

Compton's effect II

Primary photon (known f_1)

$E_1 = hf_1$

Moving electron after the collision (it stays in the matter undetected)

$E_2 = hf_2;\ E_2 < E_1$

Photon after the collision (can be detected)

Electron with mass m at rest before the collision with photon

$$\Delta\lambda = \frac{h}{mc}(1-\cos\Theta) \qquad (5)$$

Wave-Particle Duality
the Principle of Complementarity

- There is a group of experiments showing that light are waves and another group, the examples of which we have just seen, supports the particle theory of light. This dilemma is referred as the wave-particle duality.
- To clarify the situation Niels Bohr proposed his principle of complementarity: To understand any given experiment, we must use either the wave or the photon theory, but not both. Yet we must be aware of both theories should we have a full understanding of light. Therefore these two aspects of light complement one another.

De Broglie waves I

- In the year 1923 Luis de Broglie (1892-1987) came with, at this time a brave idea, that the wave particle-dualism is a common property of micro-world and it is symmetric, so that waves exhibit sometimes particle properties and particles can be attributed (now de Broglie) wavelength which is a function of their linear momentum :

$$\lambda = \frac{h}{p}$$

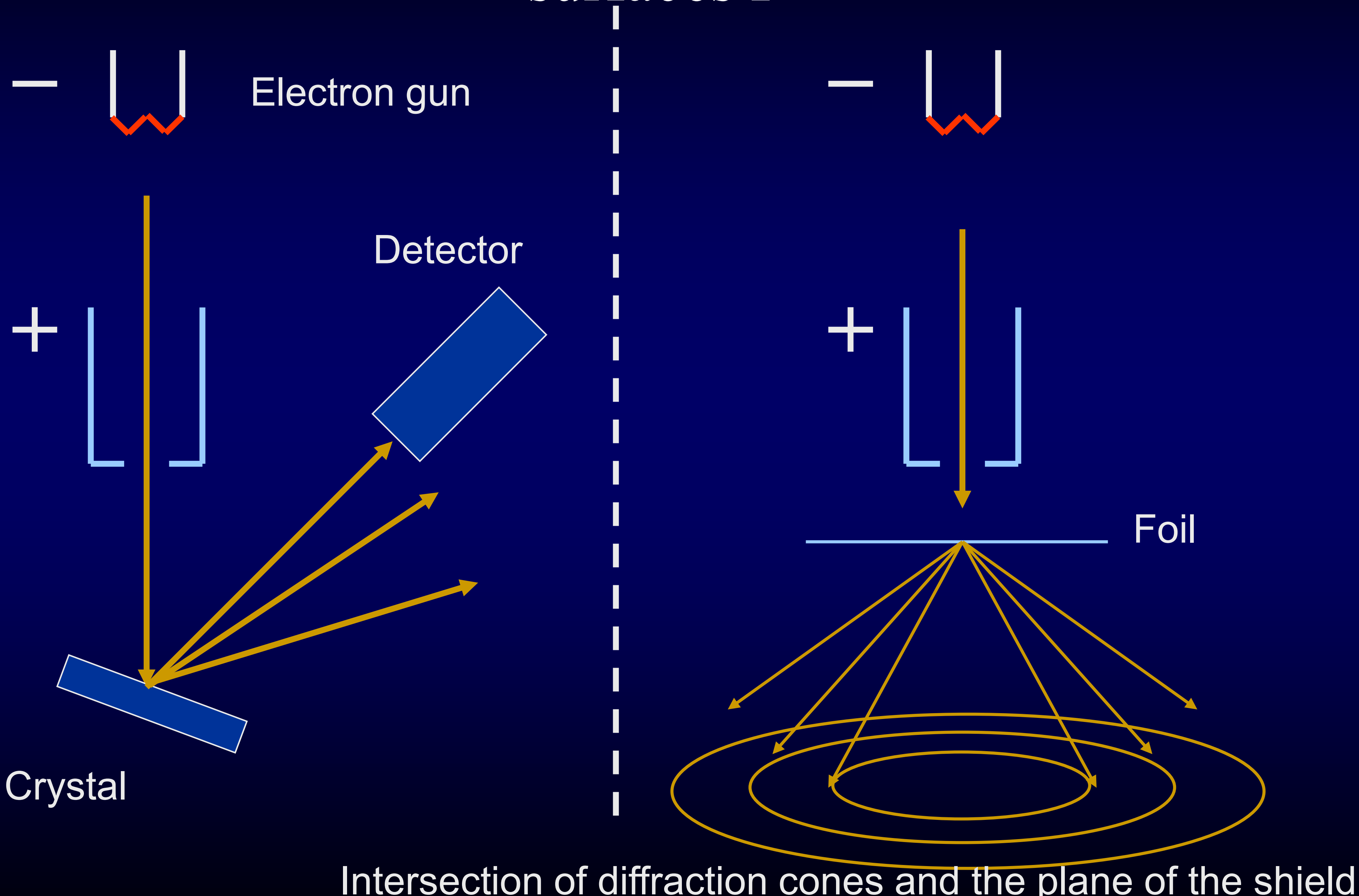
De Broglie waves II

- It comes from the analogy with photons, where $E = hf$ and $m_0 = 0$, which from the STR leads to $E = cp = hf$.
- Obviously, waves corresponding to macroscopic bodies are immeasurably (so far?) short but in the micro-world it is different :
  - Running Beatle (100 kg, 10 m/s) $\lambda \cong 10^{-37}$ m
  - Running Beetle (0.001 kg, 1 cm/s) $\lambda \cong 10^{-29}$ m
  - Electron ($9.1 \cdot 10^{-31}$ kg, $1 \cdot 10^{6}$ m/s) $\lambda \cong 10^{-10}$ m

De Broglie waves III

- Following discoveries proved that De Broglie had been right.
- E.g. the circumference of every stable orbit in Bohr's model of the hydrogen atom is integer multiple of De Broglie's wavelength.
- Soon after De Broglie's hypotheses had appeared electron diffraction would be discovered. Since the wavelength of electrons can be tuned to be of the order of atomic bond length the method is important for structure analysis.
- Wave properties of electrons are also important for construction of electron microscopes and accelerators.

Electron diffraction on crystals and thin surfaces I



Electron diffraction on crystals and thin surfaces II

The wavelength of an electron beam :

$$\frac{1}{2}mv^2 = eU \Rightarrow v = \sqrt{\frac{2eU}{m}} \Rightarrow p = mv = \sqrt{2meU}$$

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meU}}$$

for $U = 8.10^3$ V

$$\lambda = \frac{6{,}626.10^{-34}}{\sqrt{2.9{,}11.10^{-31}.1{,}602.10^{-19}.8.10^3}} = 1{,}37.10^{-11}\,\mathrm{m}$$

The Bohr model I

- Another important problem was to explain the existence the discrete lines in atomic spectra. The wavelength for the first then known Balmer's series was :

$$\sigma = \frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$$

$n = 3, 4 \ldots$ a $R = 1.0974.10^7\ m^{-1}$ is so called Rydberger's constant.

The Bohr model II

- Later two more series were discovered and for all a single equation holds :

$$\sigma = \frac{1}{\lambda} = R(\frac{1}{k^2} - \frac{1}{n^2})$$

$n = k+1, k+2, k+3$ ...

  - in UV region $k = 1$ Lyman's
  - in VIS region $k = 2$ Balmer's
  - in IR region $k = 3$ Pashen's

The Bohr model III

- At the time the existence of electrons and atomic nuclei was known but the problem was how these are composed in an atom. The simple planetary model doesn't work since if electrons move on a closed orbit their motion is accelerated and they emit energy and during several picoseconds they would fall on the nuclei.
- Bohr merged the planetary model with Planck's quantum hypothesis.

The Bohr model IV

- Bohr postulated that certain orbits and corresponding energetic levels are stationary. Then atoms emit or absorb energy possibly as photons only during transition between these states :

$$hf = E_k - E_l$$

- Energy states which he found matched spectra of hydrogen and hydrogen-like atoms (Z):

$$E_n = E_1 \frac{Z^2}{n^2}$$

- The energy $-E_1 = -13.6\ eV$ is the well known energy of the ground state of hydrogen H.

X-Rays I

- In the year 1895 Wilhelm Conrad Röntgen (the 1st awardee of the Nobel prize in 1901) discovered emission of X-Rays - an effect conceptually inverse to the photoelectric effect:
- When high energy electrons fall onto matter electromagnetic radiation is emitted . Its wavelength is of the order 10^{-10} *m*. X-Ray radiation has two components. White bremsstrahlung and discrete which is analog to VIS spectrum.

X-Rays II

- The wavelength of X-Ray is comparable with the atomic bond length so it is very important for the X-Ray structure analysis.
- X-Rays are important also for the methods of X-Ray spectroscopy, which study absorption and emission spectra and also several special methods (EXAFS…).

LASER = Light Amplication of Stimulated Emission of Radiation I

- The discovery of lasers was a great breakthrough into many fields of science.
- Lasers are sources of electromagnetic radiation in many wavelength regions (IR, VIS, UV, RTG…), which is of can be :
  - collimated
  - monochromatic
  - intense
  - coherent

Laser II

- Laser (Light Application of Stimulated Emission of Radiation) is based on the effect of stimulated emission: When an photon with a suitable energy interacts with and excited atom it stimulates an emission of another photon which is its exact copy.
- Through a choice of suitable materials inverse population of excited electrons in some metastable state for sufficiently long time can be reached and the emission can be then started.

Laser III

- Usually laser has elongated shape. One cap is a normal the other a semitransparent mirror, plane or concave. Due to these mirrors photons return many times into the excited medium. This triggers an avalanche effect in the axis and narrows its spectrum.
- The medium can be a transparent crystal or gas, how it is e.g. in HeNe laser. Excitation can be done by illumination or chemically..
- Recently also semiconductor lasers with wide and important usage develop quickly.

Kirchhoff's law I

The validity of Kirchhoff's law can be verified experimentally or by a simple reasoning: Let's have a slab the two large surfaces of which are made of different materials *I* and *II*.

Near to the surface *I* we place second slab made of material *II* and is connected with a thermometer and next to surface *II* on the other side we place a third slab made of material *I*, equipped also with a thermometer.

After certain time equilibrium when all temperatures are the same is reached.

.

Kirchhoff's law II

If ε_i are emission and α_i absorption coefficients then this must hold :

$$\varepsilon_I \alpha_{II} = \varepsilon_{II} \alpha_I$$

Then the emission coefficient is always proportional to the absorption coefficient :

$$\frac{\varepsilon_I}{\alpha_I} = \frac{\varepsilon_{II}}{\alpha_{II}}$$

If, for instance, the surface I is black $\alpha_I \simeq 1$ and the surface *II* partly reflects $\alpha_{II} < 1$ then also $\varepsilon_I > \varepsilon_{II}$.

This reasoning is valid even for any wavelength.

.

Heat radiation – example

Let's have a ceramic cup with *ε = 0.7* and a stainless steel cup with *ε = 0.1*. In each of them there is *0.75 l* of tea with *95° C*. Estimate what power is irradiated from each of them to the surroundings with the temperature *20° C* ?

Assume, each cup is a cube with the edge of *10 cm* and it absorbs and emits at the same time.

$$P = \varepsilon \sigma S(T_1^4 - T_2^4) =$$

$$\varepsilon \cdot 5.67 10^{-8}\, 5 10^{-2} (368^4 - 293^4) =$$

$$\varepsilon \cdot 30 W$$

Then the ceramic cup irradiates *21 W*. The stainless cup only *3 W* so in this one the tea stays hot for longer time. But heat conductivity should also be included in!

.

Wien's law – example I

Let's estimate the surface temperature of the Sun. The maximum of distribution of the intensity is $\lambda_m \cong 500\ nm$, it lays in the visible region and it is green :

$$T = \frac{2.9010^{-3}\ m\,K}{50010^{-9}\ m} = 5800\,K \approx 6000\,K$$

There is quite an interesting question: why green plants reject the wavelength with the highest energy?

./.

Wien's law – example II

The temperature of the fiber and composition of the internal gas of incandescent bulbs is designed according to their planned use: 2200 °C and vacuum up to 25 W, 2600 °C for the common filled with a mixture of Ar & N_2 and 3000 °C for special halogen e.g. for projectors.

Tungsten is a selective emitter which in the visible region emits more than would correspond to its temperature. What would the peak λ_m be if the fiber of a normal bulb would be a blackbody?

$$\lambda_m = \frac{2.9010^{-3}}{2875} = 1009\, nm$$

So λ_m lays in the IR region and there also most of energy is irradiated. Heat radiation feels good.

./.

Wien's law – example III

How would a star with a surface temperature *32500 K* look like?

$$\lambda_m = \frac{2.9010^{-3}}{3.2510^{4}} = 89.2\,nm$$

Here λ_m lays in the UV and the star would be bluish white.

.

Compton's effect I

X-Ray beam with the wavelength *0.14 nm* is scattered on a block of graphite. What is the wavelength of the radiation scattered to the angles *0°*, *90°*, a *180°*?

The scattered wavelength is :

$$\lambda' = \lambda + \frac{h}{m_e c}(1 - \cos\Theta)$$

The quantity $\frac{h}{m_e c} = 2.43\,pm$ has a dimension of length and it is called Compton's wavelength. Here we have :

$$\lambda' = 140 + 2.43\,(1 - \cos\Theta)\ [pm]$$

So a) $\lambda' = 140\ pm$ b) $\lambda' = 142.4\ pm$ c) $\lambda' = 144.86\ pm$

.

Compton's effect II

In the interaction of a photon and electron the energy and linear momentum is conserved :

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + E_k$$

Kinetic energy E_k has to be evaluated relativistically :

$$E_k = E - E_o$$

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + E - E_o \Longrightarrow$$

$$E = \sqrt{p^2c^2 + E_o^2} = \frac{hc}{\lambda} - \frac{hc}{\lambda'} + E_o$$

Compton's effect III

Linear momentum is conserved in the (specular) plane of scattering, in the original direction of a photon x-axis and perpendicular y-axis :

$$x: \qquad \frac{h}{\lambda} = \frac{h}{\lambda'}\cos\phi + p\cos\Theta$$

$$y: \qquad 0 = \frac{h}{\lambda'}\sin\phi - p\sin\Theta$$

After rearranging, squaring and adding :

$$p^2\cos^2\Theta = \left(\frac{h}{\lambda}\right)^2 + \left(\frac{h}{\lambda'}\right)^2\cos^2\phi - \frac{2h^2}{\lambda\lambda'}\cos\Theta$$

$$p^2\sin^2\Theta = \left(\frac{h}{\lambda'}\right)^2\sin^2\phi$$

.

Compton's effect IV

$$p^2 = (\frac{h}{\lambda})^2 + (\frac{h}{\lambda'})^2 - \frac{2h^2}{\lambda\lambda'}\cos\Theta$$

We square the energy conserving equation :

$$p^2c^2 + E_0^2 =$$

$$(\frac{hc}{\lambda})^2 + (\frac{hc}{\lambda'})^2 + E_0^2 - \frac{2h^2c^2}{\lambda\lambda'} + \frac{2hcE_0}{\lambda} - \frac{2hcE_0}{\lambda'}$$

.

Compton's effect V

We cancel E_0 and substitute for the square of momentum :

$$(\frac{hc}{\lambda})^2 + (\frac{hc}{\lambda'})^2 - \frac{2h^2c^2}{\lambda\lambda'}\cos\Theta =$$

$$(\frac{hc}{\lambda})^2 + (\frac{hc}{\lambda'})^2 - \frac{2h^2c^2}{\lambda\lambda'} + \frac{2hcE_0}{\lambda} - \frac{2hcE_0}{\lambda'}$$

After rearranging :

$$2hcE_0(\frac{1}{\lambda} - \frac{1}{\lambda'}) = \frac{2h^2c^2}{\lambda\lambda'}(1-\cos\Theta)$$

.

Compton's effect VI

we substitute for the electron's rest energy $E_0 = m_0c^2$ and arrange :

$$\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)\lambda\lambda' = \frac{h}{m_0 c}(1 - \cos\Theta)$$

Finally we get the famous Compton's formula :

$$\Delta\lambda \equiv \lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\Theta)$$

.

Example – photoeffect I

Cesium layer with the work function $W_o = 1.93\ eV$, is illuminated from the distance $r = 3.5\ m$ by the light of sodium discharge lamp the strongest line of which has the wavelength $\lambda = 590\ nm$, with the power $P=100\ W$. The dimension of an electron are not known but effective radii with respect to certain effects are defined and for the interaction with a photon it is the Thompson's radius $r_e = 5.10^{-11}\ m$.

How long it would take before an electron to receive enough energy to be emitted it the energy flow was homogeneous??

What is the mean time interval between a photon flies twice through the cross-section of an electron?

The scattering cross-section of the electron is :

$$S_e = \pi r_e^2 = 7.85 \cdot 10^{-21} m^2$$

Example – photoeffect II

The energy of emitted photon in [J] is :

$$E_f = \frac{hc}{\lambda} = \frac{6.63 \cdot 10^{-34} Js \cdot 3 \cdot 10^8 ms^{-1}}{590 \cdot 10^{-9}} = 3.37 \cdot 10^{-19} J$$

The energy of emitted photon in [eV] is :

$$E_f = \frac{hc}{\lambda e} = 2.107 eV$$

Number of photons emitted in *1 s* into all angles if we expect all energy is emitted on one wavelength with 100% effectivity :

$$n_f = \frac{P}{E_f} = \frac{100W}{3.37 \cdot 10^{-19} J} = 2.97 \cdot 10^{20} s^{-1}$$

Example – photoeffect III

Intensity - power through a unit of surface in the location of the sample :

$$I = \frac{P}{4\pi r^2} = \frac{100W}{4\pi (3.5)^2 m^2} = 0.65 W m^{-2}$$

Number of photons passing through a unit of surface in the location of the sample in *1 s* :

$$n_0 = \frac{n_f}{4\pi r^2} = \frac{2.97 \cdot 10^{20} s^{-1}}{154 m^2} = 1.93 \cdot 10^{18} s^{-1} m^{-2}$$

Example – photoeffect IV

After multiplying by the electron's cross-section we get energy absorbed by electron in the unit of time :

$$P_e = IS_e = 5.1 \cdot 10^{-21} Js^{-1} (= W)$$

and number a photons through this cross-section in *1 s* :

$$n_e = n_0 S_e = 0.015 s^{-1}$$

Now we easily find out that the time necessary to accumulate the energy W_0 is around 1 minute :

$$t = \frac{W_o}{P_e} = \frac{1.93\, eV \cdot 1.6 \cdot 10^{-19} J / eV}{5.1 \cdot 10^{-21} Ws^{-1}} \cong 60\, s$$

Example – photoeffect V

Mean time of one photon crossing through the electron's cross-section is :

$$t_f = \frac{1}{n_e} = 66.1\,s$$

These times may seem comparable. The photoeffect is, however, much faster $\sim 10^{-9}\ s$. This may be explained only by dense bodies of energy - photons in the beam cross-section.

There is no way how the time necessary to suck the energy continuously could be shortened!

The Bohr Model I

Bohr accepted the planetary model but postulated only certain stationary states, characterized by certain values of the angular momentum :

$$L \equiv m_e v r_n = n\hbar \qquad\qquad n = 1, 2, \ldots$$

In planetary model the centripetal force is accomplished by the Columb's force. If we substitute for v^2 in the denominator :

$$\frac{m_e v^2}{r_n} = \frac{kZe^2}{r_n^2} \Rightarrow r_n = \frac{kZe^2}{m_e v^2} = \frac{kZe^2 m_e^2 r_n^2}{m_e n^2 \hbar^2}$$

The Bohr Model II

After rearranging we find out that radius of any orbit of any atom can be found as a multiple of Bohr's radius, which is the radius of the smallest orbit in Hydrogen.

$$r_n = \frac{\hbar^2 n^2}{ke^2 m_e Z} = r_1 \frac{n^2}{Z}$$

Similarly the energy of any orbit can be expressed using the ground energy of Hydrogen, which is the energy of the orbit closest to the nucleus.

$$E_n = \tfrac{1}{2} m_e v^2 - \frac{kZe^2}{r_n} = -\frac{k^2 e^4 m_e}{2\hbar^2} \frac{Z^2}{n^2} = E_1 \frac{Z^2}{n^2}$$

The Bohr Model III

Let's evaluate the first four orbitals of the Hydrogen :

n	r_n [pm]	E_n [eV]
1	53	-13.6
2	212	- 3.4
3	417	- 1.5
4	848	- 0.85

For a particular atom energies grow quadratically.

Radius of corresponding orbits in higher Z atoms is smaller.

Energies of bound electrons is always negative. For ionisation this or higher energy has to be absorbed by the electron.

Energy levels quadratically pack in the direction of zero energy.

Energies of absorbed or emitted photons must correspond to the transitions between the appropriate energetic states.

The Bohr Model IV

Manipulations that leads to the formula for E_n :

We substitute for $m_e v^2$ into the total energy :

$$E_n = \tfrac{1}{2} m_e v^2 - \frac{kZe^2}{r_n} = -\frac{kZe^2}{2r_n}$$

And we substitute for $1/r_n$:

$$E_n = -\frac{kZe^2 kZe^2 m_e}{2\hbar^2 n^2} = -\frac{k^2 e^4 m_e}{2\hbar^2}\frac{Z^2}{n^2} = E_1 \frac{Z^2}{n^2}$$

De Broglie Waves

For photons we use Einstein's equation for the total energy. De Broglie's great idea was that what is valid for photons could be valid also for other micro-particles :

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \Rightarrow$$

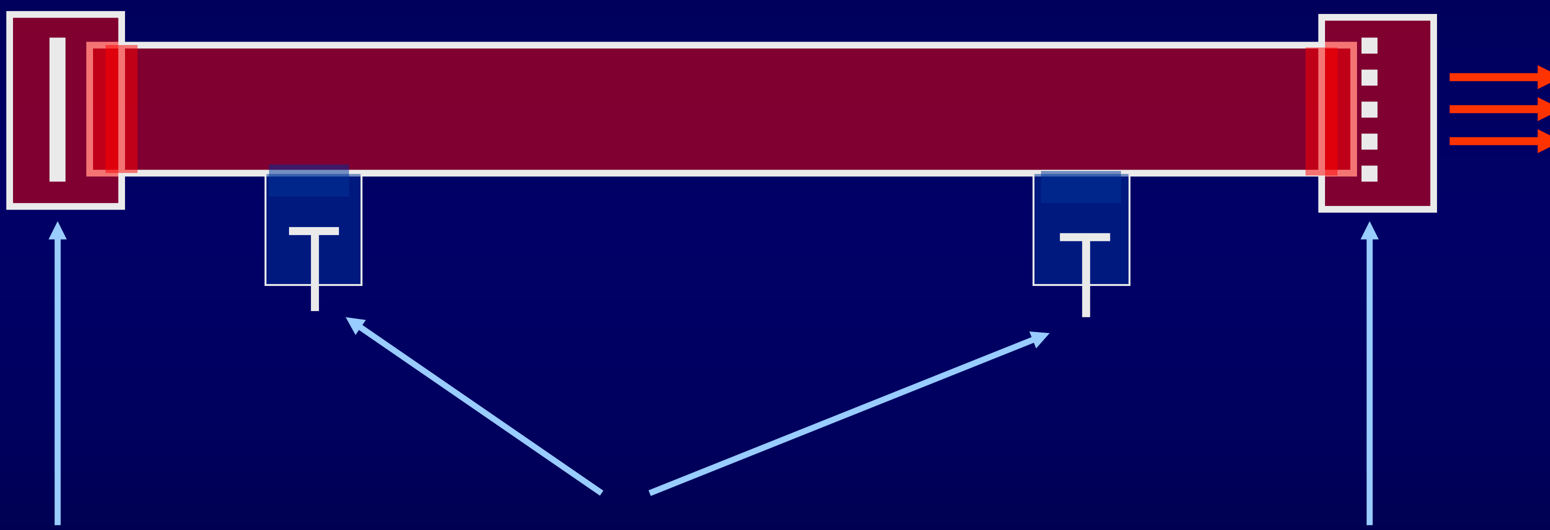
$$m^2 c^6 - m^2 u^2 c^4 = m_0^2 c^6 \Rightarrow$$

$$E^2 = p^2 c^2 + E_0^2 \Rightarrow$$

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}$$

Helium Neon Laser

Gas laser in which the active medium is a mixture of He and Ne gases.



Mirror

Electrodes to ignite the discharge

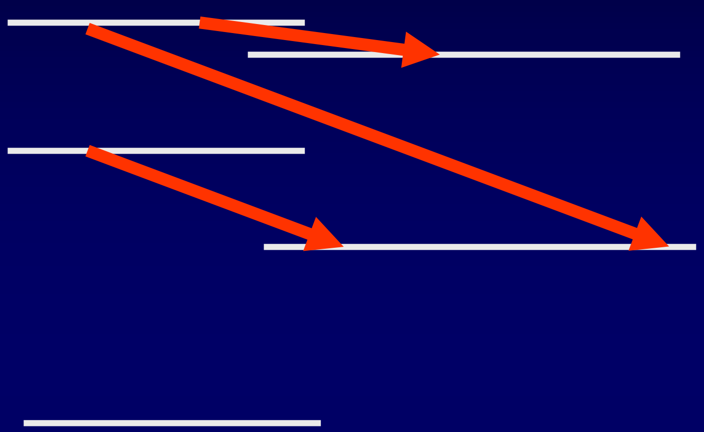
Semi-transparent mirror

Active energetic levels in the He - Ne laser

3s - 3p ….3,391 μm

3s - 2p ….0,633 μm

2s - 2p ….1,152 μm



Physics

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INSTITUTE OF APPLIED PHYSICS AND MATHEMATICS

Physics

Illustrative Problems and Solutions

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Problem 01: The side of an elementary cube in the primitive cubic crystal is a = 1.2 Å. How many atoms are there in 1 cm^3? (Avogadro number $N_A = 6.023\ 10^{23}\ mol^{-1}$)

Solution: 1D in 1 cm: $1/1.2\ 10^{-8} = 8.33\ 10^7 \rightarrow$ 3D in 1 cm^3: $(8.33)^3 = 5.79\ 10^{23}$

Problem 02: The street system in a town consists of radial roads and rings. If we are entering via one radial road and plan to leave by another, how should we drive to take the shortest path?

Solution: This depends on the angle φ between the two radial roads. The distance through the center is always $2R$ so for $\varphi < 2$ radians it is shorter to take the ring.

Problem 03: The highest and most tilted point on the Leaning Tower of Pisa is 56.7 m high and its projection lays 3.9 m from the bottom of the wall. How is the tower tilted and what is its real length?

Solution: $\mathrm{tg}\varphi = y/x$; $x = h \cos\varphi$; $y = h \sin\varphi$; $\varphi = \mathrm{atg}(y/x) = 1.5$ rad = 86.063˚. The tilt from the vertical direction is only 3.94˚. The length of the tower is $h = y/ \sin\varphi$ = 56.8 m. The tilt is relatively small but human eye is sensitive to the tilt for the vertical or horizontal directions.

Problem 04: Test basic vector operations using these vectors: $\underline{a}$ = (2, -1, -2); $\underline{b}$ = (6, -3, 1); $\underline{c}$ = (-2, 1, -5). For instance, is $\underline{c}$ perpendicular to $\underline{a}+\underline{b}$?

Solution: $(\underline{a}+\underline{b}).\underline{c}$=(8, -4, -1).(-2, 1, -5)= -16-4+5=-15. These vectors are not perpendicular.

Problem 05: A car travels 50 km by the speed of 60 km/h and the next 50 km by the speed of 120 km/h. What is its average speed?

Solution: Using a primitive average (60+120)/2 is wrong! The proper definition must be employed: $\langle v \rangle = \Sigma s_i / \Sigma t_i$ = 100*60/75 = 80 km/h

Problem 06: As could be seen from another boat flowing with the current our boat travels with a velocity v_l = 1.85 m/s to the North perpendicular to the current. Tom standing on a shore sees the current to flow with the velocity v_p = 1.2 m/s to the West. What is the velocity of our boat seen by him?

Solution: He sees it traveling to the North-West under an angle φ to the West. Vectors of the velocity of the boat and the velocity of he current are perpendicular, so $\varphi = \mathrm{arctg}(v_l/v_p)$ = arctg(1.54) = 57.03˚. v=2.2051 m/s

Problem 07: The coordinate of a particle can be described by the equation $x(t) = 2.1\ t^2 - t + 2.8$.

a) What kind of motion is it?
b) After how long is the speed zero?
c) What is the speed after 5 s?
d) What is the coordinate after 3 s?
e) What is the average speed between 3 and 5 s?

Solution: ab) $v(t) = \mathrm{d}x/\mathrm{d}t = 4.2t–1$, $a(t) = \mathrm{d}v/\mathrm{d}t = 4.2$, $v(t_0) = 4.2\ t_0–1=0$ uniformly slowing up to t_0=1/4,2=0,2381s then uniformly accelerated.
cd) $v(5) = 4.2*5–5 = 9.5$ m/s c) $x(3) = 2.1*9–3+2.8 = 18.7$ m; $x(5) = 50.3$ m.
e) $\langle v \rangle = (x(5) – x(3))/(5-3) = 15.8$ m/s.

Problem 08: Two cars start from two places 100 km apart approaching each other by the speeds 20 ms^{-1}, respectively 30 ms^{-1}. Where and when they meet? Try also graphical solution.

Solution: The equations of their motion $x_a(t)=20t$; $x_b(t)=100000-30t$; Their solution use the fact that if they meet it has to be at the same time and in the same point, so both equations must hold together $t(x_a=x_b)=2000$ s

Problem 09: The driver of a car driving by a speed of 108 km/h notices a police radar ahead. He starts to brake with the acceleration of $a = –2{,}5\ ms^{-2}$. When his car really starts to slow down the radar is just 100 m ahead. What speed will be measured?

Solution: From the formulas for $x(t)$ and $v(t)$ we eliminate time: $v_2 = v_1 + a(t_2 - t_1) \rightarrow (t_2 - t_1) = (v_2 - v_1)/a \rightarrow x_2 = x_1 + v_1(t_2 - t_1) + a/2\ (t_2 - t_1)^2 \rightarrow d = x_2 - x_1 = v_1(v_2 - v_1)/a + a/2\ ((v_2 - v_1)/a)^2 = ½\ a\ (v_2^2 – v_1^2) \rightarrow v_2^2 = 2ad + v_1^2 = 72$ km/h

Problem 10: Estimate how fast must the air-bag inflate to help a driver driving at the speed of 108 km/h, which stops at 1 m after crushing?

Solution: The average acceleration is $a = (v_2^2 – v_1^2)/2d = -450ms^{-2} \rightarrow \mathrm{d}t = (v_2 – v_1)/a = 0.0666$ s. It is particularly the acceleration what kills!

Problem 11: What are the period T, the frequency f, the angular speed ω and the circumference speed v of points on the equator and on the 50th and 60th parallel?

Solution: T=86400s; R=6378km; Equator: O_E=40000km; v_E=1666km/h=463m/s; At the parallel φ: $O(\varphi)=O_E\cos(\varphi)$; $O(50)=O_E\cos50$=40000*0.463=25712; $v(50)$=1071km/h=298m/s; Earth parameters: Tsiderial=23h56m4.09s=86164s; tropics ~23.27°; polar circles ~66.33°

Problem 12: An axis at rest starts to rotate with the constant angular acceleration $\varepsilon = 2s^{-2}$. After how long the centripetal acceleration reaches four times the value of the tangent acceleration? Is it necessary to know the diameter?

Solution: It is the case of uniformly accelerated rotation so $a_t=\varepsilon r$ and $a_c=\varepsilon^2t^2r \rightarrow \varepsilon^2t^2r=4\varepsilon r \rightarrow t^2=4/\varepsilon$

Problem 13: The crank arm of a bicycle has the length r = 171 mm, the front gearwheels have from 22 to 54 teeth and rear freewheels have from 9 to 36 teeth. That allows for a great span of gear ratios. What is the distance the rear wheel of diameter D = 696 mm travels at one rotation of the pedal crank arm when the gear ratio 1:1 and 54:9? How high would the ‚High bicycle‘ have to be?

Solution: The circumference speed of the wheels connected by a chain is constant. Moreover the circumference of a gearwheel is proportional to its number of teeth as well as its radius or the diameter.: Let ω be the angle velocity of pedals and the front gearwheel and Ω the angle velocity of the freewheel and the rear wheel and z, Z the respective number of teeth then $\omega z=\Omega Z$!; Since z is proportional to the respective radius or circumference $\rightarrow \Omega=\omega z/Z$; For the ratio 1:1 $\Omega=\omega$; $p=R/r=2.03$ and the distance $L_1=2.187$m. For the ratio 54:9 $\Omega=6*\omega$; $p=zR/Zr=12.2$; $L_1=13.1$m. In this case the ‚High bicycle' would have to have the diameter DD=4.2m!

Problem 14: A little puppet with the mass of m = 100 g is hanging on a string on the mirror of a car.

a) What is its force on the mirror at rest?

b) Later the car uniformly accelerates on a horizontal road so the puppet is inclined by 6° from the vertical. What is the acceleration of the car?

c) What is the force on the mirror now?

Solution: The string is stretched always in the direction of the resulting force.

a) At rest it is vertical and the force is the weight G=mg=1 N;

b) $a/g=\mathrm{tg}\varphi \rightarrow a=g.\mathrm{tg}\varphi\approx g.\varphi=1.05$ m/s^2;

c) Since here the force of gravity and inertia are perpendicular we can use Pythagoras law $G\approx>1$N.

Problem 15: A child on a merry-go-round rotates on a circle of the radius 5 m and the string that holds him is inclined 30° from the vertical. What is the angular speed of the merry-go-round?

Solution: $\mathrm{tg}\varphi = a_c/g = \omega^2 r/g \rightarrow \omega = \mathrm{sqrt}(g.\mathrm{tg}\varphi/r)$

Problem 16: A body with the mass of m = 2 kg is hanging on a string over a pulley. The influence of the pulley and string can be neglected. We act on the string by a force. What happens if

a) $F = 0$,

b) $0 < F < mg$,

c) $F = mg$

d) $F > mg$?

e) Can the acceleration bigger than the free-fall g be reached?

Solution: We use the concept of dynamic equilibrium $F+F^*=mg \rightarrow$ The inertial force $F^* = ma_b = mg-F \rightarrow a_b=(mg-F)/m$

a) $F=0 \rightarrow a_b=g$ Free-fall.

b) $0<F<mg$ $a_b=(mg-F)/m<g$ slowed free-fall.

c) $F=mg \rightarrow a_b=0$. The body moves uniformly or is in rest.

d) The body accelerates upward $a_d<0$.

e) This can't be reached applying force on a string over a pulley. The force has to act directly on the body, be vertical and (its component) point downward.

Problem 17: Two bodies hang on a string over a pulley. The influence of the pulley and string can be neglected. The body on the left has the mass m_1 = 3 kg, the body on the right has the mass m_2 = 2 kg. How the system moves when the string is released? What if there was some initial speed?

Solution: Since there is a non-zero resulting force of gravity the system is uniformly accelerated. If the influence of the pulley, bearings and the string can be neglected then the force stretching the string on both sides of the pulley are the same. So in analogy with the previous problem we can write for the left side $m_1g=m_1a+F$ and for the right $F=m_2g+m_2a$ after excluding F we get the acceleration $a=(m_1- m_2)/(m_1+m_2)$. The heavier body accelerates downward the other upward. If the initial speed it opposite to the acceleration the system first slows down, stops and then moves the same way as it would move from the rest.

Problem 18: A toy railway has three wagons with masses m_1 = 10 kg, m_2 = 5 kg, m_3 = 5 kg and an engine which pulls the first wagon behind it by the force of 40 N. The first wagon pulls the second and that pulls the third one. The rails are horizontal. How will the train move? What are the forces that pull the individual wagons if we neglect all resistance forces?

Solution: The total mass is m=20 kg so the uniform acceleration of the train is $a=F/m$=2 m/s^2. If a force F_i acts on the i-th wagon a part of it accelerates this wagon and F_{i+1} pulls the rest of the train. Generally we can write $F_{i+1}=F_i - a.m_i$. So the forces are F_1=40N, F_2=20N and F_3=20N.

Problem 19: A body with the mass of m = 2kg is pulled from the rest by a constant horizontal force F = 40N on a horizontal plane. How it moves if

a) the friction can be neglected?

b) The coefficient of friction is μ = 0.2?

Solution: a) $a_a=F/m$;

b) $a_b=(F-\mu mg)/m$ Friction acts always opposite to the speed. This has to be taken into account if there is some initial speed.

Problem 20: A body with the mass of m_1 = 3kg is at rest on a plane inclined from the horizontal direction by α = 30˚. It is connected by a string over a pulley with another body with the mass of m_2 = 2kg which hangs freely. The influence of the pulley and string can be neglected. How will the system move after being release if

a) the friction can be neglected?

b) the coefficient of friction is μ = 0.1?

Solution: If the string and pulley can be neglected the force in the string is everywhere the same.

a) Only the component of weight parallel with the inclined plane has to be taken into account.

$$m_1 g \sin\alpha = m_1 a_a + F_{La};\ F_{La} = m_2 g + m_2 a_a \Rightarrow a = g\frac{m_1 \sin\alpha - m_2}{m_1 + m_2} = -1ms^{-2}$$

b) The hanging body prevailed the body on the inclined plane. This is not the general case but when considering friction this can't change. In the worst case the resulting force will not be enough sufficient and the system stays at rest.

$$m_1 g \sin\alpha + \mu m_1 g \cos\alpha + m_1 a_b = F_{Lb};\ F_{Lb} + m_2 a_b = m_2 g \Rightarrow$$
$$a = g\frac{m_2 - m_1(\sin\alpha + \mu\cos\alpha)}{m_1 + m_2} = 0.48\,ms^{-2}$$

Problem 21: A cheap mallet consists of a metal part with a shape of a cylinder of the diameter 10 cm and mass m_1 = 1.5 kg and a wooden handle a much thinner cylinder 20 cm long with the mass of m_2 = 0.5 kg. The handle is glued to the perimeter of the metal part in its middle so the rotation axes are perpendicular. Where is the center of mass of this mallet?

Solution: The general formula for the center of mass of a system of several particles is

$$\vec{r}_C = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

Formally the same formula can be used for the center of mass of a body consisting of smaller parts if we know their masses and centers of mass.
Due to the symmetry our problem is one dimensional. We can coincide the *x*-axis with the axis of the handle and place the origin into its centre of mass. Then:

$$x_C = \frac{0.5\cdot 0 + 1.5\cdot 15}{0.5+1.5} = \frac{22.5}{2} = 11.25\,cm$$

The center of mass is in the metal part 11.25 cm from the c.o.m. of the handle.

Problem 22 (Torque): When we have to use a scale with unequal arms we can find the correct weight by measuring twice. When we put the body of unknown mass m to the left we have to balance it by the weight of m_1 = 330 g on the right. When we put the unknown body to the right we have to balance it by the weight m_2 = 920 g on the left. What is the unknown mass m?

Solution: The scales compare torque of the bodies on the left and on the right which try to rotate it in opposite sense. If the length of the left arm is a and of the right one b we can write: $mga = m_1gb$ and $m_2ga = mgb \Rightarrow m$ = sqrt(m_1m_2 = 551g) – geometrical average.

Problem 23: Peter whose mass is m_P = 60 kg is standing at rest on a 'long-board' with a mass of m_B = 20 kg and Paul is standing on a ground. Suddenly Peter starts to run on his board. What will be the velocity of the board when Peter reaches the speed of v_P = 4.2 m/s relatively to Paul?

Solution: If any external influence can be neglected the total linear momentum of the system Peter-board must be constant, in our case zero. So: $0 = p_P + p_B = m_Pv_P + m_Bv_B \Rightarrow v_B = -v_Pm_P/m_B = -12.6\ ms^{-1}$. The minus sing means obviously that the board has an opposite velocity than Peter!

Problem 24: Peter whose mass is m_P = 60 kg is standing at rest on the circumference of a circular disk, that can rotate around its axis, with the diameter d = 6 m and the angular momentum J = 1800 kgm^2 and Paul is standing on a ground. Suddenly Peter starts to run on the disk along the circumference. What will be the angular speed of the disk when Peter's velocity reaches v_P = 4.2 m/s relatively to Paul?

Solution: If any external influence can be neglected the total angular momentum of the system Peter-disk must be constant, in our case zero. So: $0 = b_P + b_d = mr^2(v_P/r) + J\omega_d \Rightarrow \omega_d = -mrv_P/J = -0.42\ rads^{-1}$. The minus sing means obviously that the sense of the rotations of the disk is opposite to the sense of Peter's motion!

Problem 25: A body of the mass $m_1 = 3$ kg hangs on a string from a pulley. On the other side hangs another body with the mass of $m_2 = 2$ kg. The influence of the string and resistance in bearings can be neglected and the bodies are held at rest. What will be the motion after the system is released?
a) If the influence of the pulley can be neglected?
b) If the pulley is a cylinder with the mass of $m_3 = 1$ kg and $r = 10$ cm?

Solution: In either cases the heavier body will move down uniformly accelerated.
a) If the influence of the pulley can be neglected then the stress forces on both sides are equal:

$$m_1 g = m_1 a_a + F_{S1} \wedge F_{S2} = m_2 g + m_2 a_b \wedge F_{S1} = F_{S2} \Rightarrow a_a = g\frac{m_1 - m_2}{m_1 + m_2} = 2ms^{-2}$$

b) The weight of the heavier mass must also rotate the pulley:

$$m_1 g = m_1 a_b + F_{S1} \wedge F_{S2} = m_2 g + m_2 a_b \wedge r(F_{S1} - F_{S2}) = J\varepsilon \Rightarrow a_b = g\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}m_3} = 1.82ms^{-2}$$

Problem 26: A star with the mass of twice that of our Sun and a radius $r_1 = 7.10^5$ km rotated with the period of 10 days ($T_1 = 864000$ s). What will be the frequency of rotation of it has collapsed to $r_2 = 10$ km?

Solution: If we suppose that the mass change can be neglected the angular momentum conserves:

$$J_1\omega_1 = J_2\omega_2 \Rightarrow f_2 = \frac{J_1}{T_1 J_2} = \frac{r_1^2}{T_1 r_2^2} = \frac{4\ .1^{1}\ \ 9^{0}0}{8.6\ .1^{7}\ \ 4\ \ 0} = 5.6\ .1^{3} H7$$

Problem 27: Adam with the mass of $m_A = 75$ kg sits on o bench at the distance 50 cm from Beata with the mass of $m_B = 50$ kg. What is the force of gravity between them? What is the acceleration?

Solution: The force of gravity attracts all bodies that have non-zero mass. Forces acting on Adam and Beata have the same magnitude and opposite direction. From Newton's law: $|F|=1.10^{-6}$ N. But the acceleration depends on the mass and is not the same $a_p=1.334.10^{-8}$ ms^{-2}; $a_b=-2.10^{-8}$ ms^{-2}.

Problem 28: The gravitational acceleration is $g = 9.83$ m/s^2. Estimate the mass and density of the Earth if $R_E = 6378$ km?

Solution: The gravitational acceleration near the surface of the Earth is equal to the **intensity** of the field of gravity. So we use the Newton's law:

$$mg = \frac{m\kappa M}{R_E^2} \Rightarrow M = \frac{gR_E^2}{\kappa} = 6.10^{24} kg \Rightarrow$$

$$\rho = \frac{gR_E^2}{\kappa \frac{4}{3}\pi R_E^3} = \frac{3g}{4\kappa\pi R_E} = 5.5.10^3\ kgm^{-3}$$

Problem 29: The gravitational acceleration on the surface of our Moon? What would be the difference between the body on the Earth and the Moon?

Solution: The scales measure the force of gravity. Usually it is used on the Earth so it can be calibrated directly in mass units. $\kappa = 6.67.10^{-11}$ Nm^2kg^{-2}; $M_M = 7{,}342.10^{22}$ kg; $R_M = 1{,}737.10^6$ m

$$mg_M = \frac{\kappa M_M m}{R_M^2} \Rightarrow g_M = \frac{\kappa M_M}{R_M^2} = 1.62\,ms^{-2} \Rightarrow \frac{g}{g_M} \cong 6$$

Problem 30: A sphere of the mass of m = 10 kg is pushed the distance s = 20 m up the plane inclined from the horizontal direction by the angle 30º. What work has to be done and what potential energy the body gains? Compare cases
a) without friction
b) with the friction coefficient of $\mu = 0.1$?

Solution: a) If friction can be neglected the potential energy gain is equal to the work $\Delta E_p=F.s= s.G.\sin\alpha=Gh=mgh=1$ kJ.
b) In the real case part of the work is changed to the thermal energy by friction and dissipated. So the work must be higher by $\Delta W=s.F_d=s.\mu.G.\cos\alpha$=20.0.1.100.sqrt(3/4)=173J.
So the work 1173 J is done but only 1000 J can be in principle used back.

Problem 31: We compress a spring with the stiffness of k=100 N/m from the equilibrium by s = 1 cm.
a) What is the average force we have to put in?
b) What work has to be done?
c) What potential energy will the spring gain?

Solution: A spring has a special shape so that even relatively large deformations stay elastic and a simple direct proportionality relation between the force and deformation holds $F(x)=kx$.
a) The average force is $\langle F\rangle=(F(x_2)-F(x_1))/2=0.5$N.
b) We can use the average force $W=\langle F\rangle s=5$ mJ but more general and elegant is to integrate: $dW=kx\,dx \Rightarrow W=kX^2/2=5$ mJ.
c) If loses can be neglected the deformation work is equal to the gain of the potential energy.

Problem 32: Where and at what distance from the Earth surface must the geostationary satellite move? What is its potential energy?

Solution: The geostationary satellite has to rotate around the Earth axis with the same angular velocity. Since it is also a satellite it must rotate in a equator plane. When $\kappa = 6.67.10^{-11}$ Nm^2kg^{-2};

$$R = R_Z + h \Rightarrow \frac{mv^2}{R} = m\omega^2 R = \frac{\kappa M_Z m}{R^2} \Rightarrow$$
$$R = \sqrt[3]{\frac{\kappa M_Z T^2}{4\pi^2}} = 4.22310^7\,m \Rightarrow h = 35.65.10^3\,km$$
$$\Delta E_P = -\kappa M_Z\left(\tfrac{1}{R} - \tfrac{1}{R_Z}\right)m = m\cdot 5.13\cdot 10^7\,J$$

Problem 33: The shot has the mass of m = 7.5 g. From the barrel 120 mm long it leaves with the speed of 390 m/s.
a) What is its kinetic energy and what was the average force that acts on it during its flight through a barrel? Part of the energy of the gun-powder is dissipated and the shot also.
b) How high it would climb if it was shot perpendicularly up?

Solution: a) The kinetic energy $E_k=mv^2/2=570\text{J}=s.F_p \Rightarrow F_p=E_k/s=4.75$ kN.
b) If loses are neglected we use the conservation of energy $Ep=Ek \Rightarrow h=v^2/2g=7605\text{m}$

Problem 34: The shot from the previous problem hits the ballistic pendulum with the mass $M = 1$ kg and stays in it. What will be the displacement angle?

Solution: If the shot stays in the pendulum the linear momentum is conserved. Then we use conservation of kinetic and potential energy: $V=mv/(m+M)=2.9\text{m/s} \Rightarrow h=V^2/2g=0.42\text{m}$.

Problem 35: When the Cu anode of an X-Ray tube is heated and electron is released with negligible kinetic energy. Then it is accelerated towards the anode by a voltage difference of 100 kV. What kinetic energy and what speed it gains?

Solution: From the definition of potential $E_k=E_p=q_eU=100\text{keV}=1.6.10^{-19}.10^5=1.6.10^{-14}\text{J}$. The speed calculated classically $v=1.88.10^8\text{m/s}$. It is almost 2/3 of c so the relativistic formula $E=mc^2$ must be used as a basis for accurate calculation.

Problem 36: What is the ratio of the submerged volume of an iceberg with the density $\rho_i = 920 \text{ kg.m}^{-3}$ in a see water with the density of $\rho_w = 1025 \text{ kg.m}^{-3}$?

Solution: Let V_i be the volume of the iceberg and $V_w=kV_i$ volume of water displaced by the iceberg. According to Archimedes principle $V_i \rho_i = kV_i \rho_w = V_w \rho_w = m.$ so the ratio is $k = \rho_i / \rho_w = 920/1025 = 0.9$

Problem 37: A balloon with the weight $m = 1800$ kg is descending with the acceleration $a = 0.5 \text{ ms}^{-2}$. What load has to be released to stop the descending?

Solution: Should the descending be stopped, the buoyant force must be equal to the new weight : $F_v = mg - ma = (m - \Delta m)g \Rightarrow \Delta m = \dfrac{ma}{g} = 90kg$

Problem 38: Hiero II, the king of Syracus ordered Archimedes to find out whether his new crown is really golden. Archimedes found out that the crown weights $m = 14.7$ kg in the air and in water with the density $\rho_V = 1.10^3 \text{ kg/m}^3$ weights $m_v = 13.4$ kg and knew that the density of gold is $\rho_{Au} = 19300 \text{ kg.m}^{-3}$. Is the crown golden?

Solution: $V = m/\rho_x$; $V = m_v/(\rho_x - \rho_V) \Rightarrow \rho_x = \rho_V\, m/(m-m_v) = 11308 \text{ kgm}^{-3}$. The crown is probably lead. The double-weighing method is used to find out density of stuff that has irregular shape.

Problem 39: What is the total force acting one side of an aquarium? The side is $h = 1$ m high and $w = 3$ m wide.

Solution: Pressure of water depends on the current depth, so we have to integrate by strips dx high and w wide:

$$F = \int_0^h \rho g x w dx = \tfrac{1}{2}\rho g w [x^2]_0^h = \frac{\rho g w h^2}{2} = 15000N$$

Problem 40: An airplane flying horizontally has the mass of m = 2.10^6 kg. The area of its wings is S = 1200 m^2 and the density of air is ρ_0 = 1.2928 kg/m^3. The speed of air flow around the lower surface of the wings is v_L = 100/s. What must be the speed of the flow around the upper surface v_U?

Solution: The pressure difference acting on the lower resp. upper surface of the wings must be equal to the weight of the plane. Since the plane flies horizontally we use Bernolli's equation with the potential energy terms cancelled:

$$\frac{\rho v_L^2}{2} + p_L = \frac{\rho v_U^2}{2} + p_U \wedge S(p_L - p_U) = mg \Leftrightarrow \tfrac{\rho}{2}(v_U^2 - v_L^2) = \Delta p = \frac{mg}{S} \Rightarrow$$
$$v_U = \sqrt{v_L^2 + \frac{2mg}{\rho S}} = 151.3\, m/s$$

Problem 41: Water flows down from a faucet of the cross section of S_1 = 1.2 cm^2. A 45 mm lower the cross section decreases to S_2 = 0.35 cm^2. What is the flow speed from the facet v_1 and what is the flow-rate Q? Try to outline the general dependence.

Solution: We expect that there is atmospheric pressure around the stream:

$$\frac{\rho v_1^2}{2} + \rho g h_1 + p_1 = \frac{\rho v_2^2}{2} + \rho g h_2 + p_2 \wedge p_1 = p_2 \wedge v_2^2 = \frac{v_1^2 S_1^2}{S_2^2} \Rightarrow$$
$$v_1 = S_2\sqrt{\frac{2g(h_1 - h_2)}{S_1^2 - S_2^2}} \Rightarrow Q = v_1 S_1 = S_1 S_2 \sqrt{\frac{2g(h_1 - h_2)}{S_1^2 - S_2^2}}$$

v_1=0.286 m/s, Q=34 ml s^{-1}

Problem 42: Water flows vertically up the fountain jet with the cross section of S_1 = 4 cm^2 with the flow-rate of Q_1 = 1 L/s. What will be the cross section S_2 and flow speed v_2 Δh=20 cm above the jet?

Solution: The exactly same equation as in the previous problem can be used: v_1=2.5 m/s, v_2=1.5 m/s, S_2=6,66cm^2

Problem 43: How long L_1 can a steel string hanging vertically be so it doesn't break by its own weight?
b) How long L_2 it can be if its cross section is S = 3 mm^2 and there is a load of m = 60 kg on its end? The density of steel is ρ = 7.8.10^3 kg/m^3 and a peek stress σ_{max} = 3.14.10^8 Pa?

Solution: We can assume the string breaks close to its upper end since there all the weight is supported and derive a formula for the second more general case:

$$\sigma_{MAX} > \frac{\rho L S g + mg}{S} \Rightarrow L_1 = \frac{\sigma_{MAX}}{\rho g};\ L_2 = \frac{S\sigma_{MAX} - mg}{\rho S g}$$

L_1=4026m, L_2=1461m

Problem 44: The steel string is L = 100 m long, its density is ρ = 7.8.10^3 kg/m^3 and Young's modulus E = 2.10^{10} Pa. It hangs vertically by one of its ends. What is its prolongation by its own weight?

Solution: Since there is different stress along the length we have to integrate. Let define vertical axis x with the origin at the free end of the original un-deformed string. A piece of the length be dx is prolonged by d$l(x)$ that depends on the weight of the string below. We assume to be in the range of proportional deformation. So Hook's law holds and for d$l(x)$ we get :

$$E\frac{dl(x)}{dx} = \sigma(x) \Rightarrow \frac{dl(x)}{dx} = \frac{\rho gSx}{ES} = \frac{\rho gx}{E} \Rightarrow dl(x) = \frac{\rho gxdx}{E}$$

So the total prolongation ΔL we get by integration i.e. by adding pieces d$l(x)$ over the original length of the string:

$$\Delta L = \int_0^L dl(x) = \frac{\rho g}{E}\int_0^L xdx = \frac{\rho gL^2}{2E} = 1.91cm$$

Problem 45: A pump of a house hot-water heating system is placed in the basement. It pumps water under a pressure of 3 atm, at a speed of 0.5 m/s through a pipe with the inner diameter of 4 cm. What will be the pressure and speed in a pipe with the inner diameter of 2.6 cm on the second floor 5 m above?

Solution: Let's assume the pipes are not branched. Then from the principle of continuity:

$$v_2 = v_1\frac{S_1}{S_2} = v_1\frac{d_1^2}{d_2^2} = 1.18\,m/s$$

We use Bernoulli's equation with h_1=0

$$\frac{\rho v_1^2}{2} + p_1 = \frac{\rho v_2^2}{2} + \rho gh_2 + p_2 \Rightarrow p_2 = p_1 + \tfrac{\rho}{2}(v_1^2 - v_2^2 - 2gh_2) = 2.5\cdot 10^5\ Pa$$

Problem 46: Water level is 10 cm above the bottom in a cylinder with the cross section of S_1 = 50 cm^2. How long it takes since half of the volume leaves by a hole at the bottom with the cross section of S_2 = 5 mm^2?

Solution: We define a vertical y axis with the origin at the bottom of the cylinder. Although $S_1>>S_2$ in this problem we can't neglect motion of the level v_1:

$$v_1 = \frac{-dy}{dt} \Rightarrow dt = \frac{-dy}{v_1} \Rightarrow t = \int_h^{h/2}\frac{-dy}{v_1} = \int_{h/2}^{h}\frac{dy}{v_1(y)}$$

Quite contrary we have to find out its dependence on the current level $v_1(y)$.

$$S_1v_1 = S_2v_2 \Rightarrow v_2^2 = \frac{S_1^2}{S_2^2}v_1^2 = k\,{}_1^2 \Rightarrow \tfrac{1}{2}\rho v_1^2 + \rho g \;= \tfrac{1}{2}\rho k\,{}_1^2 y \Rightarrow \frac{1}{v_1} = \sqrt{\frac{k-1}{2g}} \Rightarrow$$

$$t = \int_{h/2}^{h}\frac{d}{v_1} = \sqrt{\frac{k-1}{2g}}\int_{h/2}^{h}\frac{yd}{\sqrt{y}} = 2\sqrt{\frac{k-1}{2g}}\left[\sqrt{y}\right]_{h/2}^{h} = 2\sqrt{\frac{k-1}{2g}}\left(\sqrt{h} - \sqrt{h/2}\right) = \sqrt{\frac{(k-1)h}{g}}\left(\sqrt{2} - 1\right) \cong 4\ \ s$$

Problem 47: A homogeneous disk of the mass of m = 3.4 kg and diameter d = 60 cm can rotate around a horizontal axis fixed to a point at its circumference. Find the period T of its oscillations and its reduced length?

Solution: This is typical case of a physical pendulum:

$$\omega=\sqrt{\frac{mg}{J}}\Rightarrow T=2\pi\sqrt{\frac{Jd}{mg}}=2\pi\sqrt{\frac{\lambda}{g}}\Leftarrow\lambda=\frac{J}{am}$$

$$a=r\wedge J_T=\tfrac{1}{2}m\,{}^{2}\Rightarrow J_O=\tfrac{3}{2}m\,r^2\Rightarrow\lambda=\tfrac{3}{2}r=\tfrac{3}{4}d\Rightarrow T=2\pi\sqrt{\frac{3d}{4g}}=1.3\ s$$

Problem 48: A whale sends ultrasonic impulses with the frequency of f = 200 kHz. The compression modulus of water is $B = 2.10^9$ Pa and its density ρ= 1025 kg/m^3.
a) What is the speed of these (sound) waves?
b) What is their wavelength? How long it takes till the signal returns from an obstacle which is at a distance of d = 100 m?

Solution: $c=\sqrt{\frac{B}{\rho}}=1.4.10^{3}\,m\,s^{-1};\ \lambda=\frac{c}{f}=7.10^{-3}\,m;\ \tau=\frac{d}{c}=0.1\ s$

Problem 49: Our Sun emits light whose peak intensity is in the visible region at around 500 nm. How does this wavelength change in diamond if the refraction index of a diamond is n = 2.419.

Solution: When radiation passes through an interface of two media its frequency stays constant while the speed of the waves and thereby the wavelength may change accordingly.
$f=c/\lambda_0=v/\lambda$; λ_0=500 nm; $\lambda=\lambda_0/n$=206.7 nm. In the air that would be in UV region but in diamond it is not since even the UV waves become shorter.

Problem 50: Our Sun emits light whose peak intensity is in the visible region at around 500 nm. What is its surface temperature?

Solution: According to Wien's law T=2.9.10^{-3}/500.10^{-9}=6000K

Problem 51: A light monochromatic wave has the wavelength of λ = 650 nm. What is the energy of its photons? Could this light be absorbed by Hydrogen atom?

Solution: According to Planck law E(eV)=1240.7/650=1,91 eV. Using Bohr's the energy levels are $E(n)=-13.6/n^2$ and the first three energies: E_1=-13.6eV, E_2=-3.4eV; E_3=-1.51eV →1.9eV doesn't correspond to any of energy differences between them. If the wavelength was λ=656.3nm its energy would be 1.89 eV which corresponds to the E_2 to E_3 absorption.

Problem 52: Through a piece of conductor with the resistance of R = 5 Ω, the charge of Q = 40 C passed within the time interval τ = 16 s. What work had to be done if the current decreased uniformly to zero during this time? Who did the work and where is the corresponding energy?

Solution:

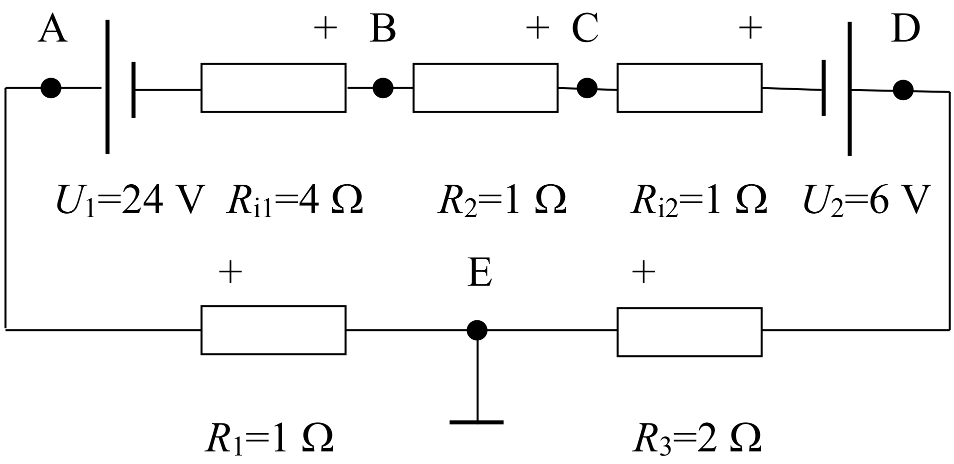
$$P(t)=\frac{dW(t)}{dt}=RI^2(t)\Rightarrow dW(t)=RI^2(t)dt$$

$$I(t)=I_0-kt\wedge I(\tau)=I_0-k\tau=0\Rightarrow I(t)=I_0(1-\frac{t}{\tau})\Rightarrow$$

$$W=RI_0^2\int_0^\tau(1-\frac{t}{\tau})^2dt=\frac{RI_0^2\tau}{3}=666.7\,J$$

This work could have be done by a specially programmed power source and has changed into the thermal energy.

Problem 53: What is the current in the following diagram? What power is brought to the circuit by each of the power sources? What is the voltage difference in the points A, B, C, D and the point E? Note that the resistors R_{i1} and R_{i2} are the internal resistors of the respective power sources.



Solution: The circuit has only one loop and the power sources are connected against each other. So since the sum of all resistors R=9Ω, the current is 2A going counter-clockwise. The voltage on each of the resistors follows Ohm's law. The polarities are given by the current orientation and are denoted by the + signs. So U_A=2V; U_B=-14V; U_C=-12V; U_D=-4V; U_E=0V. The first power source brings U_1I=48W into the circuit, U_2I=12W charges the second power source and the rest RI^2=36W is dissipated into heat on the resistors. Note that separation of a real power source into the ideal voltage and the ideal inner resistance in series is a model. The inner point of this model can't be reached from outside and doesn't even exist.

Problem 54: Between two conductor plates with the area of S = 200 cm^2 there is a slab 1 mm thick made of glass with the relative permitivity ε_{rs} = 7. On its both surfaces there is a thin layer of paraffin 0.2 mm with the relative permitivity of ε_{rp} = 2. What is the capacitance of this capacitor?

Solution: If boundary effects can be neglected the electric field lines are perpendicular to all the surfaces which are equipotential. So the total capacity is the same as the capacity of three capacitors in series. One has 1 mm of glass and the two other 0.2 mm of paraffin:

$$\frac{1}{C} = \frac{d_s}{\varepsilon_{rs}\varepsilon_0 S} + 2\frac{d_p}{\varepsilon_{rp}\varepsilon_0 S} = \frac{1}{\varepsilon_0 S}\left(\frac{d_s}{\varepsilon_{rs}} + \frac{2d_p}{\varepsilon_{rp}}\right) \Rightarrow$$

$$C = \frac{\varepsilon_{rs}\varepsilon_{rp}\varepsilon_0 S}{(\varepsilon_{rp}d_s + 2\varepsilon_{rs}d_p)} = \frac{7\cdot 2\cdot 8.86\cdot 10^{-12}\cdot 2\cdot 10^{-2}}{(2\cdot 1 + 2\cdot 7\cdot 0.2)\cdot 10^{-3}} = 516.8\cdot 10^{-12} = 516.8\,pF$$

Problem 55: A simple capacitor with two conductive plates and a slab of dielectric material with the relative permitivity $\varepsilon_r = 5$ between them has a capacity of $C_1 = 500$ pF. It is charged to 5 kV and disconnected from the power source. What work has to be done to remove the dielectric slab?

Solution: If the power source is disconnected the charge on the capacitor is conserved (unless we discharge it by bad manipulation). The capacity of a simple capacitor is directly proportional to the relative permitivity. Removing the slab decreases the capacity and increases energy so work has to be done. If the power source was not disconnected it would keep constant voltage on the capacitor and removing the slab would decrease the energy so the slab would jump out:

$$Q = UC_1 = 2.5\,\mu C;\; C_2 = \frac{C_1}{\varepsilon_r};\; E_1 = \frac{Q^2}{2C_i} \Rightarrow$$

$$W = \Delta E = \frac{Q^2}{2C_2} - \frac{Q^2}{2C_1} = \frac{Q^2}{2C_1}(\varepsilon_r - 1) = 2.5\,mJ$$

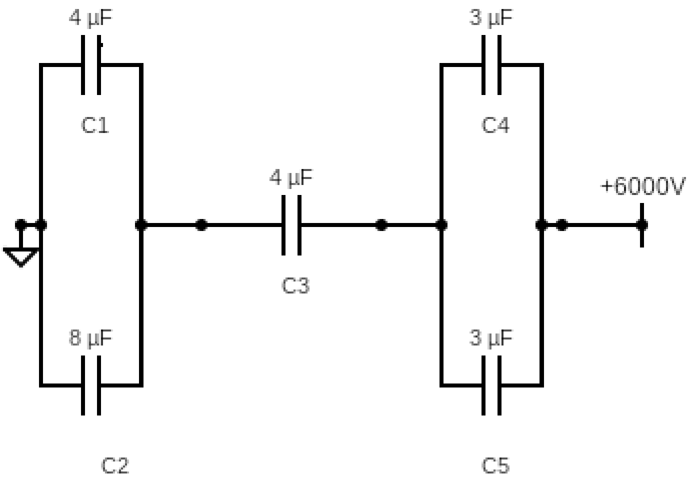
Problem 56: A capacitor consists of two conductive plates of the area $S = 500$ cm^2 1 cm apart is charged to 5 kV and disconnected from the power source. What work has to be done to increase the distance of the plates to 4 cm?

Solution: If the power source is disconnected the charge on the capacitor is conserved. The capacity of a simple capacitor is indirectly proportional to the distance of the plates. By increasing of this distance the capacitance decreases so the energy increases so work of the external agent has to be done to achieve this:

$$C_1 = \frac{\varepsilon_0 S}{d_1} = 44.3\,pF;\; Q = UC_1 = 222\,nC;\; C_2 = \frac{d_1 C_1}{d_2};\; E_1 = \frac{Q^2}{2C_i} \Rightarrow$$

$$W = \Delta E = \frac{Q^2}{2C_2} - \frac{Q^2}{2C_1} = \frac{Q^2}{2C_1}\left(\frac{d_2}{d_1} - 1\right) = 1.66\,mJ$$

Problem 57: Capacitors $C_1 = 4$ µF, $C_2 = 8$ µF, $C_3 = 4$ µF, $C_4 = 3$ µF and $C_5 = 3$ µF are connected according to the diagram. This combination is charged by the power source with the voltage $U = 6000$ V. What is the energy, charge and voltage on each of the capacitors and the whole combination?



Solution:
$C_{12}=C_1+C_2=12$ µF, $C_{45}=C_4+C_5=6$ µF, $1/C=1/12+3/12+2/12=6/12$ → $C=2$ µF → $Q=CU=12\text{mC}=Q_{12}=Q_3=Q_{45}$ → $U_{12}=1000\text{V}$, $U_3=3000\text{V}$, $U_{45}=2000\text{V}$ → $Q_1=U_{12}C_1=4\text{mC}$, $Q_2=U_{12}C_2=8\text{mC}$, $Q_4=U_{45}C_4=6\text{mC}=Q_5$. $E=QU=12\text{mC}*6\text{kV}=72\text{J}$, $E\text{i}=Q\text{i}U\text{i}=(4;8;36;12;12)\text{J}$

Problem 58: When charging an accumulator the voltage and current values are U_1 = 6.35 V and I_C = 4 A. Later when discharging the same accumulator the values change to U_2 = 5.65 V a I_D = 6 A. What are the parameters of the accumulator? What is the energy balance during charging and discharging?

Solution: Although an accumulator has many important parameters describing its capacity, self-discharging etc. Here we assume just a simple model of electromotoric voltage U_E in series with the internal resistance R_I. The voltage on the internal resistance corresponds to the direction of the current so the voltage on the terminals must be higher during charging then it is in the case of discharging. However there is an energy loss on the internal resistance in both cases.

$$U_C = U_E + I_C R_I$$
$$U_D = U_E - I_D R_I$$
$$\Rightarrow U_E = 6.07\,V;\ R_I = 0.07\,\Omega$$

During charging part of the power $U_1*I_C=25.4$ W of the power source $R_I*I^2_C=1.12$ W is dissipated and the rest conserved in the accumulator. During discharging, from the power of the accumulator $U_E*I_D=36.42$ W another part of the previously conserved energy $R_I*I^2_D=2.52$ W is dissipated. The effective storage of energy is a big issue!

Problem 59: If a proton flies horizontally to the North there is no force acting on it. If it flies vertically up with a speed of 5.10^6 m/s a force of the $F=8.10^{-14}$ N directed to the West acts to it. What is the magnitude and direction of the vector of the magnetic induction $\vec{B}$?

Solution: We use Lorentz formula $\vec{F}_m = \vec{v} \times \vec{B}$: The direction of $\vec{B}$ is horizontally to the North and the magnitude is $B=F/qv=0.1\text{T}$

Problem 60: Two kind of ions fly through the velocity filter that has magnetic induction B_1 and perpendicular electric intensity E to a main chamber of a mass spectrometer with the magnetic induction of B_2. Particles of the inner standard ^{12}C: $m_C = 12.0\ m_u$ move on a circle with the radius of $r_C = 22.4$ cm and unknown ions on a circle with $r_X = 26.2$ cm. How do these machines work? What are probably the unknown particles?

Solution: From the equilibrium in the velocity filter: $qE=F_E=F_M=qvB_1 \Rightarrow v=E/B_1$. In the mass spectrometer $mv^2/r=qvB_2 \Rightarrow m=qB_2r/v=qB_1B_2r/E \Rightarrow m_X/m_C=r_X/r_C \Rightarrow m_X=14.036\ m_u$. The unknown particles have the mass number 14, so it could be ^{14}N or ^{14}C or ^{14}O. To decide which of them would desire finer resolution and/or chemical separation/analysis

Problem 61: What will be the total voltage U on a conductive rod with the length of $L = 1.5$m that moves perpendicularly to B of the homogeneous magnetic field with the induction $B =$ 0.2 T?

Solution: A magnetic force acts on the charges within the rod pushing them to one end. This causes a build-up of electric field and the electric force acts in the opposite direction. This leads to the equilibrium: $qvB=qE=qU/L \Rightarrow U=BvL=0.6$ V

Problem 62: What will be the speed of a conductive rod of the length of $L = 1.5$ m in homogenous magnetic field with the induction of $B = 0.2$ T if a voltage of $U = 0.6$ V is connected to its ends?

Solution: A force depending on the current flowing through the rod acts on it: $F=BLI$. At the same time voltage on the speed of the rod is induced on its ends. The orientation is directed in the opposite sense then is the voltage of the power source $U_E(v)=BvL$. The total current and thereby the force depend on the speed of the rod: $I(v)=[U-U_E(v)]/R$, $F(v)=BLI(v)$. If there is no mechanical resistance on the rod it motion will reach equilibrium speed so that both the total current and total force are zero: $v=U/BL=0.6/(1.5.0.2)=2$m/s. Compare this with the previous problem!

Problem 63: What will be the total voltage U on a rotating conductive rod with the length of $L = 1.5$ m that rotates around its end with the period of $T = 0.5$ s in a homogeneous magnetic field with the induction $B = 0.2$ T?

Solution: Each piece of the rod dr moves with different speed depending on its radius of rotation r so we have to integrate over all such pieces:

$$dU = Bv(r)dr \Rightarrow U = \int_0^L Bv(r)dr = B\omega\int_0^L rdr = \frac{B\omega L^2}{2} = \frac{B\pi L^2}{T} = 2.83V.$$

Rotating generator is easier to accomplish since the homogeneous field can be relatively small.

Problem 64: Two beams of monochromatic light with the wavelength of $\lambda = 550$ nm fly in parallel and are in-phase. What will be their phase difference if one of them passes a slab with the refraction index of $n = 1.6$ and the thickness if $d = 2.6$ μm? What would change if the slab was of diamond?

Solution: $\Delta\varphi=2\pi d(n-1)/\lambda=17.821$rad;$\Delta\varphi_0=\mathrm{mod}(\Delta\varphi,2\pi)=5.255$;$n_C=2.419$;$\Delta\varphi_C=42.177$rad

Problem 65: To what depth of water with the density of $\rho_0 = 1.05\ \mathrm{gcm^{-3}}$ and temperature $t = 20\ °\mathrm{C}$ it is necessary to submerge a elastic balloon filled with Krypton $M = 85$ g/mol so it will not come out?

Solution: The density of the Krypton compressed in the balloon depends on the surrounding pressure so we have to find at what depth it reaches the density of surrounding water: $\rho(p(h))=\rho_0$: EOS: $pV=nRT=mRT/M \Rightarrow \rho(p)=p.M/RT= \rho_0 ghM/RT \Rightarrow h= (\rho/\rho_0).RT/gM$.so specially for us $h_1= RT/gM$ =2.86 km

Problem 66: A virus has a mass of 10^{-17}g. In what height will its appearance be significant at Room Temperature of $T = 300$ K?

Solution: A probability $P \sim \exp(-E_{pot}/kT)$; The concentration of the virus is significant if the probability >1/e so the exponent <1: $mgh<kT \rightarrow h<kT/mg$=41mm

Problem 67: A wagon with the mass of m_1=10 t moving with the velocity of $v_1 = 0.9$ m/s collides with a wagon with the mass of $m_2 = 30$ t at rest and the wagons connect at the moment of their collision.

a) What will be their common velocity u?

b) What would have to be the original velocity v_{2b} of the second wagon so that after the collision both wagons are at rest?

Solution: If the wagons (bodies) move together after the collision it is the case of inelastic collision.

a) $P_A=p_1=m_1v_1$; $P_P=(m_1+m_2)u_1$; $P_A=P_B \Rightarrow u_1=m_1v_1/(m_1+m_2)$=0.225 m/s

b) We want $P_P=0 \Rightarrow 0=P_A=m_1v_1+m_2v_{2b} \Rightarrow v_{2b}=-m_1v_1/m_2$=-0.3 m/s. The wagons must move in opposite directions.

Problem 68: An iron beam with the length of $L = 20$ m and the mass of $m_1 = 1500$ kg is supported at its ends by concrete pillars of the height $h = 5$ m. In ¼ of its length it supports a machine of the mass $m_2 = 15000$ kg. The maximum stress of concrete is pull is 2.10^6 Pa, in push 2.10^7 Pa and Young modulus 2.10^{10} Pa. The safety factor is sf=6.

a) What must be the minimum cross section of the pillars so that they are not overloaded?

b) What will be the deformation of the pillar with higher load?

Solution: The static equilibrium desires that the sum of all forces and the sum of all torques are zero (vectors). Let the supporting force on the left be F_1, on the right F_2 and the heavy machine closer to the right one. Apparently, there will be more load on the right pillar so we can consider just this one and the other one will be over-designed, so we can calculate the torques relatively to the left supporting point.

$$F_2L - G_1\tfrac{L}{2} - G_2\tfrac{3L}{4} = 0 \Rightarrow F_2 = \frac{2G_1+3G_2}{4} = 1 \quad 2N$$

$$F_1 + F_2 - G_1 - G_2 = 0 \Rightarrow F_1 = G_1 + G_2 - F_2 = 4 \quad 5N$$

$$\frac{F_2}{S} < \frac{\sigma_p}{6} = 2.1 \quad 7 \Rightarrow S > 6\frac{1.2.1\ ^5}{2.1\ ^7} = 0.0 \quad m^2$$

$$\frac{\Delta h}{h} = \varepsilon = \frac{\sigma_p}{6E} \Rightarrow \Delta h = \frac{h\sigma_p}{6E} = 8.3.1\ ^{-4}\ m$$

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