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1. Schrödinger eqn and particle in a box

The S.E. in 1-d is

$$\left[-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right]\psi(x,t) = i\hbar\,\partial_t\,\psi(x,t)$$

assume that the potential $V(x)=0$, and that the wave function $\psi(x,t)$ satisfies the boundary condition $\psi(0,t)=0=\psi(L,t)$. ($0\leq x\leq L$ i.e. the "box")

a.) using separation of variables, obtain the time-independent S.E. $-\frac{\hbar^2}{2m}\partial_x^2 X(x) = E\,X(x)$ for the x-dependent part of the wave function. Here E is the separation constant. (In QM it has the interpretation of the Energy of the particle.)

Solve the time-dependent part of the S.E.

1b. find the solutions to the time-independent S.E. subject to the BCs. You should find that the Energy E is quantized. What are the quantized energies?

1c. Find the solution for $\psi(x,t)$ assuming that at $t=0$ we have $\psi(x,0) = \psi_0 \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)$

2. Method of Frobenius

sometimes, it is useful to try to solve a diffeq by making a power series ansatz $y(x) = \sum_{n=0}^{\infty} a_n x^n$, plug $y(x)$ into the diffeq and find a recursion relation for the a_n. If the recursion relation can be solved one obtains the Taylor series of the solution.

Example: solve $y' + ky = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = \sum_{n'=0} a_{n'+1}(n'+1)x^{n'}$$

plug in

$$\Rightarrow \sum_{n=0}^{\infty} \left[a_n + a_{n+1}(n+1)\right] x^n = 0$$

this can only be 0 for all x if all coefficients of x^n vanish

$$\Rightarrow \quad a_{n+1} = -\frac{1}{n+1} a_n \qquad n \geq 0$$

2 cont'd Note that this relation gives all a_n recursively in terms of $a_0 \equiv a$. For example

$$a_1 = -\frac{1}{1} a_0 = -a$$

$$a_2 = -\frac{1}{2} a_1 = \frac{1}{2} a$$

$$a_3 = -\frac{1}{3} a_2 = (-\tfrac{1}{3})(-\tfrac{1}{2})(-\tfrac{1}{1}) a = (-)^3 \frac{1}{3!} a$$

$$\vdots$$

$$a_n = (-)^n \frac{1}{n!} a$$

$$\Rightarrow \quad y = a \sum_{n=0}^{\infty} \frac{1}{n!} (-)^n x^n = a \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = a e^{-x}$$

which – of course – we could have also obtained by direct integration.

2a. solve $(1-x)y' - y = 0$ by method of Frobenius

(if you can't sum the series, you can peek at the answer by directly integrating the diffeq)

2b. solve $y'' - y = 0$ (Note that here you get two independent recursion relations, one for even and one for odd n's. Thus you get 2 independent solutions like you should for a 2nd order diffeq.)