

- Do not turn this page until the told to do so.
- Hand your exam in before leaving the room.
- This exam contains 11 pages (including this cover page) and 10 problems.
- A formula sheet is attached at the back.
- This is a closed book exam. **No books, notes or calculators allowed.**
- Write all your work on these sheets. If you need more space, you may use the back of the front page; clearly indicate when you have done this.
- **Organize your work**, in a reasonably neat and coherent way.
- **Mysterious or unsupported answers will not receive full credit.**

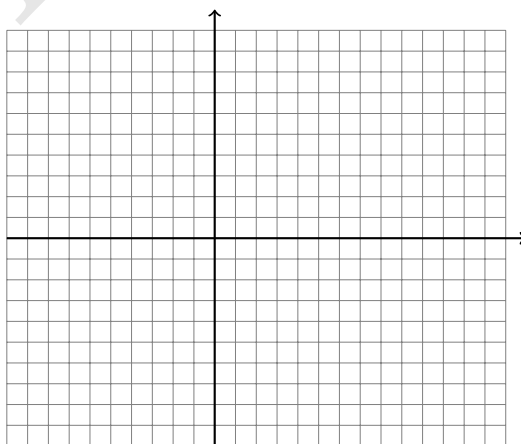
Problem	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	15	
9	10	
10	40	
Total:	100	

1. After centrifuging, the mass density of a fluid in a bucket of radius R and height L is given by (5)

$$\rho(s, \phi, z) = \rho_0 \left(1 + a \left(\frac{s}{R} \right)^2 \right)$$

in cylindrical coordinates (s, ϕ, z) , where ρ_0 and a are constants. What is the total mass M of the fluid?

2. Sketch the function $f(x) = \frac{x^2 - 4}{x^2 - 1} e^{-x}$, labeling any important qualitative features. (5)



- 3.** Express the i 'th component of the cross product $\vec{A} \times \vec{B}$ in terms of the components of $\vec{A}$ and $\vec{B}$ and the Levi-Civita symbol. (5)

- 4.** State the gradient theorem carefully. (5)

5. Prove that $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$ using index notation. (5)

6. A certain magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ where (5)

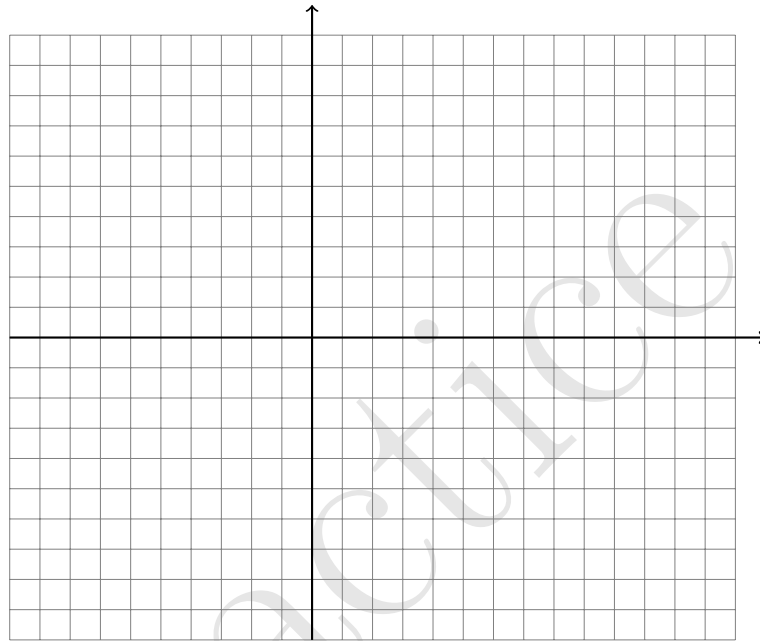
$$\vec{A} = xyz \hat{x} + x^2z \hat{y} + xyz^2 \hat{z}$$

Find a scalar $f(x, y, z)$ such that $\vec{A}' = \vec{A} + \vec{\nabla}f$ has a z -component identically zero.

7. A plane wave of light propagates in vacuum. The electric field is given by (the real part of) (5)

$$\vec{E}(\vec{r}, t) = \frac{(1+i)}{\sqrt{2}} \hat{z} e^{i(x-ct)}$$

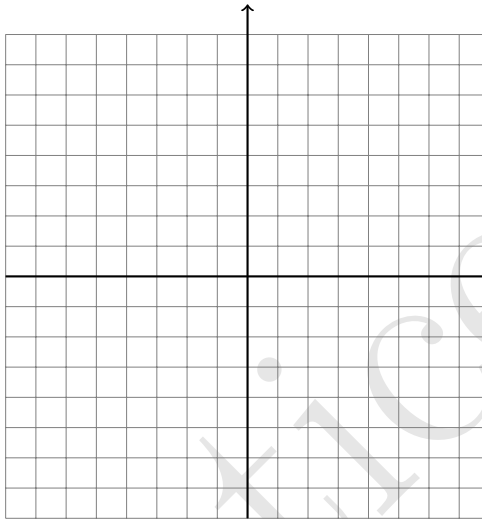
where c is the speed of light. Sketch $\vec{E}$ in the x - z plane at time $t = 0$.



8. A viscous fluid is trapped between two large parallel plates (at $z = 0$ and $z = d$). The top plate is moving at velocity $v_0 \hat{x}$. The velocity of the flow is then

$$\vec{v} = v_0 \frac{z}{d} \hat{x}$$

- (a) Sketch the flow field with a quiver plot in the XZ plane. (5)

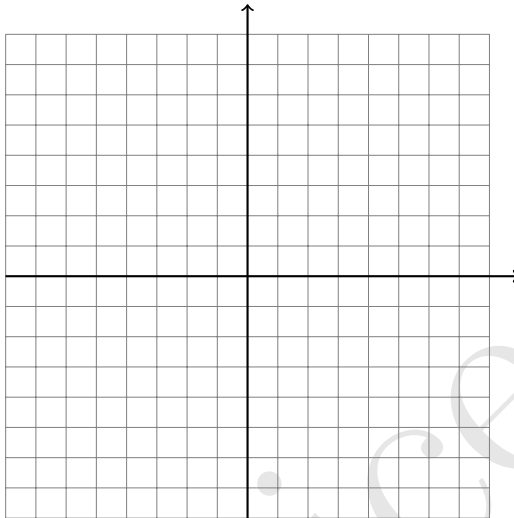


- (b) Compute the divergence $\vec{\nabla} \cdot \vec{v}$. (5)

- (c) Compute the curl $\vec{\nabla} \times \vec{v}$. (5)

9. Consider the cardioid described by $r(\theta) = 1 - \cos \theta$ in polar coordinates.

(a) Sketch the cardioid: (5)



(b) Compute the area enclosed. (5)

10. A piece of tubing with square cross section of side length a runs along the z -axis. That is, its sides lie at $x = \pm a/2$ and $y = \pm a/2$. Maple sap runs through the tube with a velocity

$$\vec{v} = v_0 \left(1 - \frac{2x}{a}\right)^2 \left(1 - \frac{2y}{a}\right)^2 \hat{z}$$

(a) Make a contour plot in the xy plane ($z = 0$) of the magnitude of the velocity field. **(5)**

(b) Make an arrow plot in the xz plane ($y = 0$) of the velocity field inside the tube. **(5)**

(c) What is $\vec{\nabla} \cdot \vec{v}$? (5)

(d) What is $\vec{\nabla} \times \vec{v}$? (5)

- (e) If the sap has mass ρ per unit volume, what mass of sap flows through the tube per unit time? (5)

- (f) Compute the circulation of $\oint_{S^2} \vec{v} \cdot d\vec{r}$ for a circle S^2 of radius $a/4$ around the origin lying in the xy plane. (5)

- (g) Compute the circulation of $\oint_R \vec{v} \cdot d\vec{r}$ for a square of side length $a/4$ lying in the xz plane, with one side on the z axis and the other at $x = a/4$ by explicit integration. (5)

- (h) Compute the same circulation as in part (g) using Stokes' theorem. (5)

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\tau = dx dy dz$

$$\text{Gradient:} \quad \nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence:} \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl:} \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$$

$$\text{Laplacian:} \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin \theta dr d\theta d\phi$

$$\text{Gradient:} \quad \nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\text{Divergence:} \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned} \text{Curl:} \quad \nabla \times \mathbf{v} &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} \\ &+ \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \end{aligned}$$

$$\text{Laplacian:} \quad \nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$

$$\text{Gradient:} \quad \nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence:} \quad \nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl:} \quad \nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$$

$$\text{Laplacian:} \quad \nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$