

| Problem | Points |
|---------|--------|
| 1       | 5      |
| 2       | 5      |
| 3       | 5      |
| 4       | 5      |
| 5       | 5      |
| 6       | 5      |
| 7       | 5      |
| 8       | 15     |
| 9       | 10     |
| 10      | 40     |
| Total:  | 100    |

1. After centrifuging, the mass density of a fluid in a bucket of radius  $R$  and height  $L$  is given by (5)

$$\rho(s, \phi, z) = \rho_0 \left( 1 + a \left( \frac{s}{R} \right)^2 \right)$$

in cylindrical coordinates  $(s, \phi, z)$ , where  $\rho_0$  and  $a$  are constants. What is the total mass  $M$  of the fluid?

$$M = \int \rho d\tau = \int_0^{2\pi} d\phi \int_0^L dz \int_0^R \rho_0 \left( 1 + a \frac{s^2}{R^2} \right) s ds$$

The  $\phi$  and  $z$  integrals give  $2\pi L$ . The  $s$  integral is:

$$\int_0^R \left( s + \frac{a s^3}{R^2} \right) ds = \frac{R^2}{2} + \frac{a}{R^2} \cdot \frac{R^4}{4} = \frac{R^2}{2} \left( 1 + \frac{a}{2} \right)$$

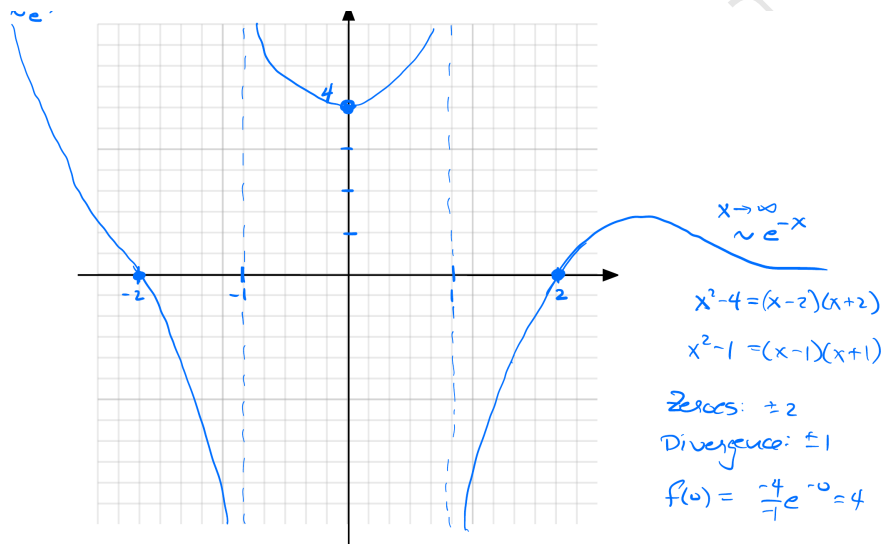
$$M = \pi L \rho_0 R^2 \left( 1 + \frac{a}{2} \right)$$

Units check:  $[\rho_0 R^2 L] = \frac{M}{L^3} \cdot L^3 = M$ . ✓

2. Sketch the function  $f(x) = \frac{x^2 - 4}{x^2 - 1} e^{-x}$ , labeling any important qualitative features. (5)

Key features to identify:

- **Zeros:**  $x^2 - 4 = (x - 2)(x + 2) = 0 \Rightarrow x = \pm 2$ .
- **Vertical asymptotes:**  $x^2 - 1 = (x - 1)(x + 1) = 0 \Rightarrow x = \pm 1$ .
- **Value at origin:**  $f(0) = \frac{-4}{-1} \cdot 1 = 4$ .
- **As  $x \rightarrow +\infty$ :**  $\frac{x^2 - 4}{x^2 - 1} \rightarrow 1$ , so  $f \sim e^{-x} \rightarrow 0^+$ .
- **As  $x \rightarrow -\infty$ :**  $\frac{x^2 - 4}{x^2 - 1} \rightarrow 1$ , so  $f \sim e^{-x} \rightarrow +\infty$ .



3. Express the  $i$ 'th component of the cross product  $\vec{A} \times \vec{B}$  in terms of the components of  $\vec{A}$  and  $\vec{B}$  and the Levi-Civita symbol. (5)

$$(\vec{A} \times \vec{B})_i = \varepsilon_{ijk} A_j B_k$$

where the Einstein summation convention (sum over repeated indices  $j$  and  $k$ ) is implied.

4. State the gradient theorem carefully. (5)

For a scalar field  $f(\vec{r})$  and a curve  $C$  from point  $\vec{a}$  to point  $\vec{b}$ :

$$\int_{\vec{a}, C}^{\vec{b}} \vec{\nabla} f \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$$

The line integral of a gradient depends only on the endpoints, not on the path.

5. Prove that  $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$  using index notation. (5)

$$\begin{aligned}\vec{\nabla} \cdot (\vec{A} \times \vec{B}) &= \partial_i (\varepsilon_{ijk} A_j B_k) \\ &= \varepsilon_{ijk} (\partial_i A_j) B_k + \varepsilon_{ijk} A_j (\partial_i B_k) \quad (\text{product rule})\end{aligned}$$

First term: relabel  $\varepsilon_{ijk} = \varepsilon_{kij}$  (cyclic permutation):

$$\varepsilon_{kij} (\partial_i A_j) B_k = (\vec{\nabla} \times \vec{A})_k B_k = (\vec{\nabla} \times \vec{A}) \cdot \vec{B}$$

Second term: swap two indices  $\varepsilon_{ijk} = -\varepsilon_{jik}$ :

$$-\varepsilon_{jik} A_j (\partial_i B_k) = -A_j (\vec{\nabla} \times \vec{B})_j = -\vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

Therefore:

$$\boxed{\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\vec{\nabla} \times \vec{B})}$$

6. A certain magnetic field  $\vec{B} = \vec{\nabla} \times \vec{A}$  where (5)

$$\vec{A} = xyz \hat{x} + x^2 z \hat{y} + xyz^2 \hat{z}$$

Find a scalar  $f(x, y, z)$  such that  $\vec{A}' = \vec{A} + \vec{\nabla} f$  has a  $z$ -component identically zero.

We need the  $z$ -component of  $\vec{A}' = \vec{A} + \vec{\nabla} f$  to vanish:

$$A_z + \frac{\partial f}{\partial z} = 0 \quad \implies \quad \frac{\partial f}{\partial z} = -xyz^2$$

Integrating with respect to  $z$ :

$$\boxed{f = -\frac{xyz^3}{3}}$$

Check:  $\hat{z} \cdot \vec{A}' = xyz^2 + \partial_z \left( -\frac{xyz^3}{3} \right) = xyz^2 - xyz^2 = 0. \quad \checkmark$

(Note:  $\vec{B}' = \vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla} f) = \vec{\nabla} \times \vec{A} = \vec{B}$ , so the physics is unchanged.)

7. A plane wave of light propagates in vacuum. The electric field is given by (the real part of) (5)

$$\vec{E}(\vec{r}, t) = \frac{(1+i)}{\sqrt{2}} \hat{z} e^{i(x-ct)}$$

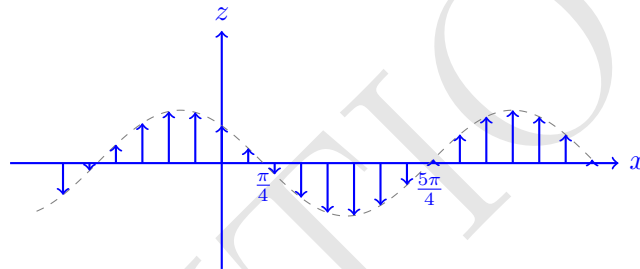
where  $c$  is the speed of light. Sketch  $\vec{E}$  in the  $x$ - $z$  plane at time  $t = 0$ .

Note that  $\frac{1+i}{\sqrt{2}} = e^{i\pi/4}$ . At  $t = 0$ :

$$\vec{E}(\vec{r}, 0) = \text{Re} \left[ e^{i\pi/4} e^{ix} \right] \hat{z} = \cos \left( x + \frac{\pi}{4} \right) \hat{z}$$

This is a **linearly polarized** wave in the  $\hat{z}$  direction, propagating in  $\hat{x}$ , with a phase shift of  $\pi/4$ .

Zeros at  $x + \pi/4 = \pi/2 + n\pi$ , i.e.  $x = \pi/4 + n\pi$ .

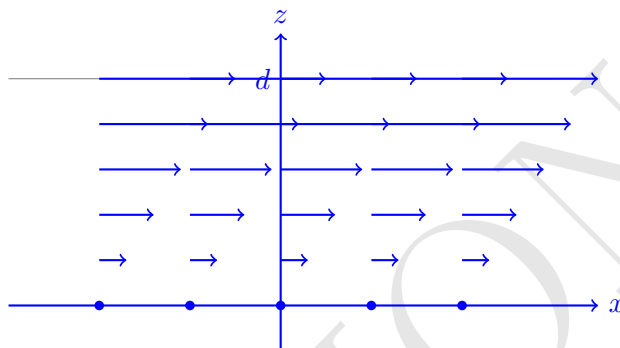


8. A viscous fluid is trapped between two large parallel plates (at  $z = 0$  and  $z = d$ ). The top plate is moving at velocity  $v_0 \hat{x}$ . The velocity of the flow is then

$$\vec{v} = v_0 \frac{z}{d} \hat{x}$$

(a) Sketch the flow field with a quiver plot in the  $XZ$  plane. (5)

All arrows point in the  $\hat{x}$  direction. The magnitude increases linearly with  $z$ : zero at the bottom plate ( $z = 0$ ) and  $v_0$  at the top plate ( $z = d$ ). The flow is independent of  $x$ .



(b) Compute the divergence  $\vec{\nabla} \cdot \vec{v}$ . (5)

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial}{\partial x} \left( \frac{v_0 z}{d} \right) + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 + 0 + 0$$

$$\boxed{\vec{\nabla} \cdot \vec{v} = 0}$$

The component  $v_x = v_0 z/d$  depends only on  $z$ , not on  $x$ , so its  $x$ -derivative vanishes.

(c) Compute the curl  $\vec{\nabla} \times \vec{v}$ . (5)

With  $\vec{v} = (v_0 z/d, 0, 0)$ :

$$(\vec{\nabla} \times \vec{v})_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 0$$

$$(\vec{\nabla} \times \vec{v})_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = \frac{v_0}{d} - 0 = \frac{v_0}{d}$$

$$(\vec{\nabla} \times \vec{v})_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$$

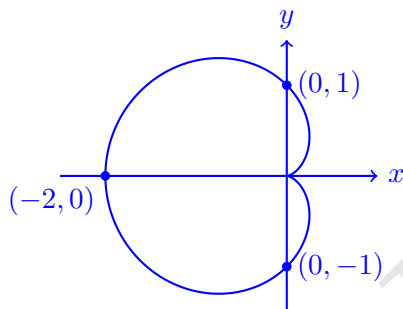
$$\boxed{\vec{\nabla} \times \vec{v} = \frac{v_0}{d} \hat{y}}$$

The curl is constant and uniform — the shear flow has constant vorticity.

9. Consider the cardioid described by  $r(\theta) = 1 - \cos \theta$  in polar coordinates. (10)

(a) Sketch the cardioid. (5)

Key points:  $r(0) = 0$  (cusp at origin),  $r(\pi/2) = 1$ ,  $r(\pi) = 2$  (farthest point, on  $-x$  axis),  $r(3\pi/2) = 1$ ,  $r(2\pi) = 0$ . The curve is symmetric about the  $x$ -axis and opens to the left.



(b) Compute the area enclosed. (5)

The area in polar coordinates is:

$$A = \int da = \int_0^{2\pi} \int_0^{r(\theta)} r \, dr \, d\theta = \int_0^{2\pi} \frac{r(\theta)^2}{2} d\theta = \frac{1}{2} \int_0^{2\pi} (1 - \cos \theta)^2 d\theta$$

Expand:

$$= \frac{1}{2} \int_0^{2\pi} (1 - 2\cos \theta + \cos^2 \theta) d\theta$$

Each term:  $\int_0^{2\pi} 1 \, d\theta = 2\pi$ ;  $\int_0^{2\pi} \cos \theta \, d\theta = 0$ ;  $\int_0^{2\pi} \cos^2 \theta \, d\theta = \pi$ .

(For the last integral:  $\int_0^{2\pi} \cos^2 \theta \, d\theta = \int_0^{2\pi} \sin^2 \theta \, d\theta$ , and their sum equals  $\int_0^{2\pi} 1 \, d\theta = 2\pi$ , so each equals  $\pi$ .)

$$A = \frac{1}{2}(2\pi + 0 + \pi) = \frac{3\pi}{2}$$

**10.** A piece of tubing with square cross section of side length  $a$  runs along the  $z$ -axis. That is, its sides lie at  $x = \pm a/2$  and  $y = \pm a/2$ . Maple sap runs through the tube with a velocity (40)

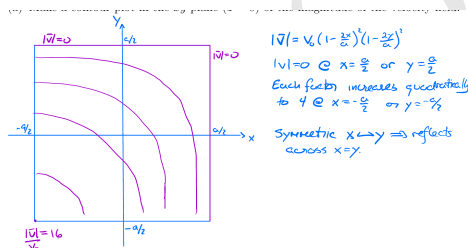
$$\vec{v} = v_0 \left(1 - \frac{2x}{a}\right)^2 \left(1 - \frac{2y}{a}\right)^2 \hat{z}$$

(a) Make a contour plot in the  $xy$  plane ( $z = 0$ ) of the magnitude of the velocity field. (5)

The magnitude is  $|\vec{v}| = v_0 \left(1 - \frac{2x}{a}\right)^2 \left(1 - \frac{2y}{a}\right)^2$ .

Key features:

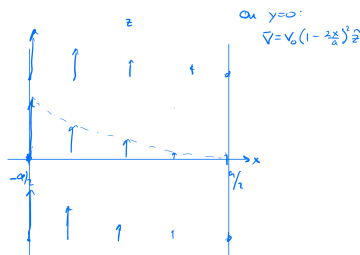
- $|\vec{v}| = 0$  along  $x = a/2$  and  $y = a/2$  (two walls of the tube).
- Maximum at  $(x, y) = (-a/2, -a/2)$ :  $|\vec{v}| = v_0 \cdot 4 \cdot 4 = 16v_0$ .
- Each factor increases quadratically from 0 at  $x = a/2$  (or  $y = a/2$ ) to 4 at  $x = -a/2$  (or  $y = -a/2$ ).
- Since  $|\vec{v}|$  factors as  $g(x) \cdot h(y)$ , the contour lines  $g(x)h(y) = c$  are symmetric under  $x \leftrightarrow y$  (reflections about the line  $x = y$ ).



(b) Make an arrow plot in the  $xz$  plane ( $y = 0$ ) of the velocity field inside the tube. (5)

On  $y = 0$ :  $\vec{v} = v_0 \left(1 - \frac{2x}{a}\right)^2 \hat{z}$ .

All arrows point in the  $\hat{z}$  direction. The magnitude is zero at  $x = a/2$ , equals  $v_0$  at  $x = 0$ , and reaches  $4v_0$  at  $x = -a/2$ . The flow is independent of  $z$ .



(c) What is  $\vec{\nabla} \cdot \vec{v}$ ? (5)

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Since  $v_x = v_y = 0$  and  $v_z$  has no  $z$ -dependence:

$$\boxed{\vec{\nabla} \cdot \vec{v} = 0}$$

The flow is incompressible.

(d) What is  $\vec{\nabla} \times \vec{v}$ ? (5)

With  $\vec{v} = (0, 0, v_z(x, y))$ :

$$(\vec{\nabla} \times \vec{v})_x = \frac{\partial v_z}{\partial y} = v_0 \left(1 - \frac{2x}{a}\right)^2 \cdot 2 \left(1 - \frac{2y}{a}\right) \left(-\frac{2}{a}\right) = -\frac{4v_0}{a} \left(1 - \frac{2x}{a}\right)^2 \left(1 - \frac{2y}{a}\right)$$

$$(\vec{\nabla} \times \vec{v})_y = -\frac{\partial v_z}{\partial x} = -v_0 \cdot 2 \left(1 - \frac{2x}{a}\right) \left(-\frac{2}{a}\right) \left(1 - \frac{2y}{a}\right)^2 = \frac{4v_0}{a} \left(1 - \frac{2x}{a}\right) \left(1 - \frac{2y}{a}\right)^2$$

$$(\vec{\nabla} \times \vec{v})_z = 0$$

Factoring:

$$\boxed{\vec{\nabla} \times \vec{v} = \frac{4v_0}{a} \left(1 - \frac{2x}{a}\right) \left(1 - \frac{2y}{a}\right) \left[ -\left(1 - \frac{2x}{a}\right) \hat{x} + \left(1 - \frac{2y}{a}\right) \hat{y} \right]}$$

- (e) If the sap has mass  $\rho$  per unit volume, what mass of sap flows through the tube per unit time? (5)

The mass flux through the cross section  $\Sigma$  (at any fixed  $z$ ) is:

$$\dot{M} = \int_{\Sigma} \rho \vec{v} \cdot d\vec{\sigma} = \rho v_0 \int_{-a/2}^{a/2} \left(1 - \frac{2x}{a}\right)^2 dx \int_{-a/2}^{a/2} \left(1 - \frac{2y}{a}\right)^2 dy$$

Each integral is the same. Substituting  $u = 1 - 2x/a$ ,  $du = -2/a dx$ :

$$\int_{-a/2}^{a/2} \left(1 - \frac{2x}{a}\right)^2 dx = \frac{a}{2} \int_0^2 u^2 du = \frac{a}{2} \cdot \frac{u^3}{3} \Big|_0^2 = \frac{a}{2} \cdot \frac{8}{3} = \frac{4a}{3}$$

$$\dot{M} = \rho v_0 \left(\frac{4a}{3}\right)^2 = \frac{16}{9} \rho v_0 a^2$$

Units check:  $[\rho v_0 a^2] = \frac{\text{kg}}{\text{m}^3} \cdot \frac{\text{m}}{\text{s}} \cdot \text{m}^2 = \frac{\text{kg}}{\text{s}}$ . ✓

- (f) Compute the circulation  $\oint_{S^2} \vec{v} \cdot d\vec{r}$  for a circle  $S^2$  of radius  $a/4$  around the origin lying in the  $xy$  plane. (5)

The circle lies in the  $xy$  plane, so  $d\vec{r}$  lies in the  $xy$  plane:  $d\vec{r} \perp \hat{z}$ .

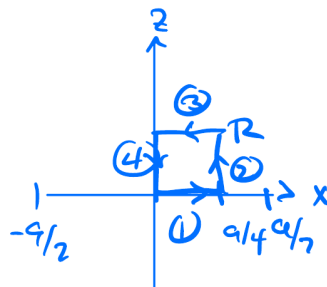
But  $\vec{v} = v_z \hat{z}$  points entirely in the  $z$ -direction.

Therefore  $\vec{v} \cdot d\vec{r} = 0$  everywhere on the circle, and:

$$\oint_{S^2} \vec{v} \cdot d\vec{r} = 0$$

- (g) Compute the circulation  $\oint_R \vec{v} \cdot d\vec{r}$  for a square of side length  $a/4$  lying in the  $xz$  plane, with one side on the  $z$  axis and the other at  $x = a/4$ , by explicit integration. (5)

- (5) (g) Compute the circulation for the same square in the  $xz$  plane, with



The square  $R$  has vertices  $(0, 0, 0) \rightarrow (a/4, 0, 0) \rightarrow (a/4, 0, a/4) \rightarrow (0, 0, a/4) \rightarrow (0, 0, 0)$ .  
On  $y = 0$ :  $\vec{v} = v_0(1 - 2x/a)^2 \hat{z}$ .

Since  $\vec{v} \propto \hat{z}$ , only the sides along  $\hat{z}$  contribute (sides ① and ③ have  $d\vec{r} \propto \hat{x}$ , so  $\vec{v} \cdot d\vec{r} = 0$ ).

**Side ②:**  $x = a/4$ ,  $z : 0 \rightarrow a/4$ ,  $d\vec{r} = dz \hat{z}$ :

$$\vec{v} = v_0(1 - \frac{1}{2})^2 \hat{z} = \frac{v_0}{4} \hat{z} \implies \int_{\text{②}} \vec{v} \cdot d\vec{r} = \frac{v_0}{4} \cdot \frac{a}{4} = \frac{av_0}{16}$$

**Side ④:**  $x = 0$ ,  $z : a/4 \rightarrow 0$ ,  $d\vec{r} = dz \hat{z}$  (with  $dz < 0$ ):

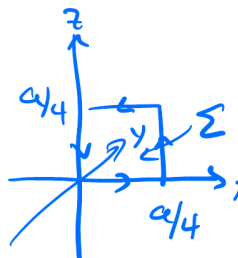
$$\vec{v} = v_0(1 - 0)^2 \hat{z} = v_0 \hat{z} \implies \int_{\text{④}} \vec{v} \cdot d\vec{r} = v_0 \int_{a/4}^0 dz = -\frac{av_0}{4}$$

Total:

$$\boxed{\oint_R \vec{v} \cdot d\vec{r} = \frac{av_0}{16} - \frac{av_0}{4} = -\frac{3av_0}{16}}$$

- (h) Compute the same circulation as in part (g) using Stokes' theorem. (5)

- (5) (h) Compute the circulation for the same square in the  $xz$  plane, with



By Stokes' theorem:

$$\oint_R \vec{v} \cdot d\vec{r} = \iint_{\Sigma} (\vec{\nabla} \times \vec{v}) \cdot d\vec{\sigma}$$

The surface  $\Sigma$  is the square in the  $xz$  plane ( $y = 0$ ). By the right-hand rule with the given traversal direction,  $d\vec{\sigma} = -\hat{y} \, dx \, dz$ .

From part (d), the  $\hat{y}$ -component of the curl on  $\Sigma$  (where  $y = 0$ ) is:

$$\hat{y} \cdot (\vec{\nabla} \times \vec{v})|_{y=0} = \frac{4v_0}{a} \left(1 - \frac{2x}{a}\right)$$

So:

$$\begin{aligned} \iint_{\Sigma} (\vec{\nabla} \times \vec{v}) \cdot (-\hat{y}) \, dx \, dz &= - \int_0^{a/4} dz \int_0^{a/4} \frac{4v_0}{a} \left(1 - \frac{2x}{a}\right) dx \\ &= -\frac{4v_0}{a} \cdot \frac{a}{4} \left[ x - \frac{x^2}{a} \right]_0^{a/4} = -v_0 \left[ \frac{a}{4} - \frac{a}{16} \right] = -v_0 \cdot \frac{3a}{16} \end{aligned}$$

$$\boxed{\oint_R \vec{v} \cdot d\vec{r} = -\frac{3av_0}{16}} \quad \checkmark \text{ (agrees with part (g))}$$