

PY 355: Methods of Theoretical Physics
Spring 2024

Midterm 1 — Solutions

Name (Print): Solutions
ID: _____

Time Limit: 75 minutes

Problem	Points	Score
1	15	
2	10	
3	20	
4	25	
5	10	
6	10	
7	20	
8	10	
Total:	120	

1. Write the first 3 non-zero terms in the Taylor series about the origin for

(5) (a) $\sinh x$

Solution: Recall $\sinh x = (e^x - e^{-x})/2$. Using $e^x = 1 + x + x^2/2! + x^3/3! + \dots$ and keeping only the odd powers:

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

(5) (b) $e^{\frac{x}{1-x}}$

Solution: First expand:

$$\frac{x}{1-x} = x(1 + x + x^2 + \dots) = x + x^2 + x^3 + \dots$$

Let $y = x + x^2 + x^3 + \dots$. Then $e^y = 1 + y + y^2/2! + \dots$:

$$\begin{aligned} e^y &= 1 + (x + x^2 + \dots) + \frac{1}{2}(x + x^2 + \dots)^2 + \dots \\ &= 1 + x + x^2 + \frac{1}{2}x^2 + \dots \end{aligned}$$

$$e^{x/(1-x)} = 1 + x + \frac{3}{2}x^2 + \dots$$

(5) (c) $\frac{1}{1-x^2}$

Solution: Geometric series with ratio $r = x^2$:

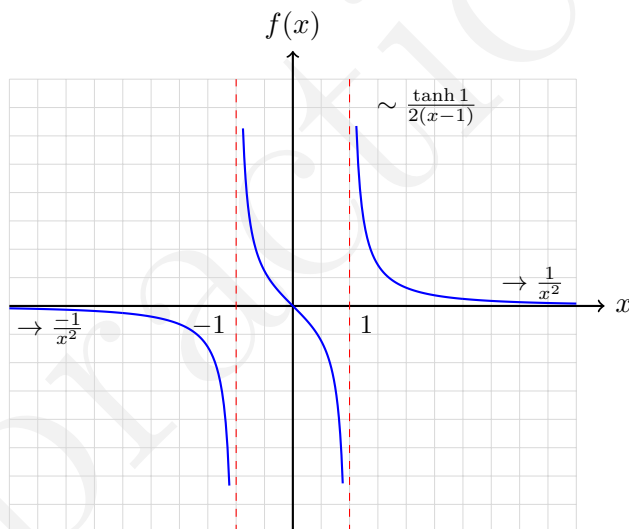
$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \dots$$

(10) 2. Sketch the real function $f(x) = \frac{\tanh x}{x^2 - 1}$ and label all important features.

Solution:

Key features:

- **Vertical asymptotes** at $x = \pm 1$ (denominator vanishes, numerator $\neq 0$).
- $f(0) = \tanh(0)/(0 - 1) = 0$. Near $x = 0$: $\tanh x \approx x$, so $f \approx x/(x^2 - 1) \approx -x$ (slope -1).
- **Odd function:** $f(-x) = -f(x)$, so the graph has point symmetry about the origin.
- As $x \rightarrow +\infty$: $\tanh x \rightarrow 1$, so $f \rightarrow 1/x^2 \rightarrow 0^+$.
- As $x \rightarrow -\infty$: $\tanh x \rightarrow -1$, so $f \rightarrow -1/x^2 \rightarrow 0^-$.
- Near $x = 1$: $x^2 - 1 \approx 2(x - 1)$, so $f \approx \tanh(1)/[2(x - 1)]$. Thus $f \rightarrow +\infty$ as $x \rightarrow 1^+$ and $f \rightarrow -\infty$ as $x \rightarrow 1^-$.
- For $-1 < x < 1$: denominator is negative, so f has opposite sign to $\tanh x$.



3. Compute the following integrals (Hint: the basic Gaussian integral over the real line is $\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\pi}$.)

(5) (a) $\int_{-\infty}^{\infty} dx x e^{-x^2}$

Solution: The integrand $x e^{-x^2}$ is an **odd function** ($g(-x) = -g(x)$), integrated over a symmetric interval.

$$\boxed{\int_{-\infty}^{\infty} dx x e^{-x^2} = 0}$$

(5) (b) $\int_{-\infty}^{\infty} dx x^2 e^{-x^2}$

Solution: Introduce a parameter a :

$$I(a) = \int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

Differentiate with respect to a :

$$I'(a) = - \int_{-\infty}^{\infty} dx x^2 e^{-ax^2} = -\frac{1}{2} \sqrt{\pi} a^{-3/2}$$

Setting $a = 1$:

$$\boxed{\int_{-\infty}^{\infty} dx x^2 e^{-x^2} = -I'(1) = \frac{\sqrt{\pi}}{2}}$$

(5) (c) $\oint_C dz e^{-z^2}$ where C is the circle of radius 1 about the origin in the complex plane

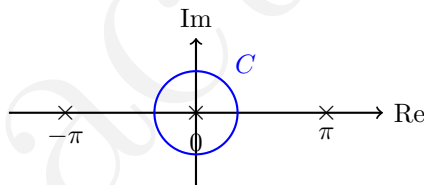
Solution: The function e^{-z^2} is **entire** (analytic everywhere—no singularities).

By **Cauchy's theorem**, the integral of an analytic function around any closed contour vanishes:

$$\oint_C dz e^{-z^2} = 0$$

(5) (d) $\oint_C dz \frac{e^{-z^2}}{\sin z}$ where C is the circle of radius 1 about the origin in the complex plane

Solution: The function $e^{-z^2}/\sin z$ has poles wherever $\sin z = 0$, i.e. at $z = n\pi$ for integer n . Only $z = 0$ lies inside the unit circle (since $\pi > 1$).



By the **residue theorem**: $\oint_C dz \frac{e^{-z^2}}{\sin z} = 2\pi i \cdot \text{Res}(z = 0)$.

Near $z = 0$: $\sin z \approx z$ and $e^{-z^2} \approx 1$, so

$$\frac{e^{-z^2}}{\sin z} \approx \frac{1}{z} \quad \implies \quad \text{Res}(z = 0) = 1$$

$$\oint_C dz \frac{e^{-z^2}}{\sin z} = 2\pi i$$

4. A block with mass m attached to a spring with spring constant k slides back and forth on a table with effective friction constant γ . Newton's law gives the equation of motion for the position $x(t)$:

$$m \frac{d^2 x}{dt^2} = -kx - \gamma \frac{dx}{dt}$$

A solution can be found by plugging in the guess $x(t) = e^{zt}$ and solving for z .

- (10) (a) Find all values of z which satisfy the equation.

Solution: Substituting $x = e^{zt}$, so $\dot{x} = z e^{zt}$ and $\ddot{x} = z^2 e^{zt}$:

$$m z^2 e^{zt} = -k e^{zt} - \gamma z e^{zt}$$

Dividing by $e^{zt} \neq 0$:

$$m z^2 + \gamma z + k = 0$$

By the quadratic formula:

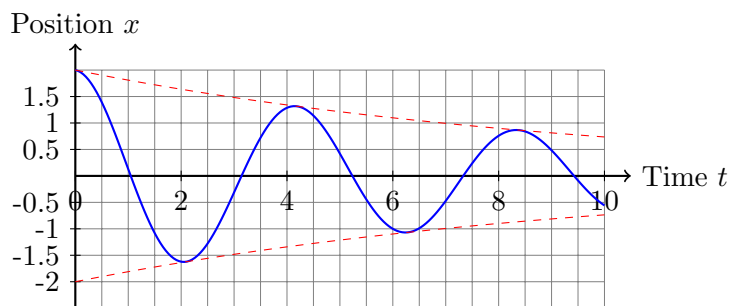
$$z_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

- (5) (b) What do the real and imaginary part of z tell you about the oscillation physically?

Solution: Writing $z = \alpha + i\omega$ with α, ω real, the solution $e^{zt} = e^{\alpha t} e^{i\omega t}$:

- **Real part** $\alpha = \text{Re}(z)$: gives the rate of exponential **decay/damping**. Since $\gamma > 0$, we have $\alpha < 0$ and the oscillations decay in time.
- **Imaginary part** $\omega = \text{Im}(z)$: gives the **angular frequency** of oscillation.

- (10) (c) Somebody comes along and kicks the mass at time $t = 0$. The resulting motion is shown in the figure, and given by (the real part of) Ce^{zt} . Determine the complex numerical values of the constants C and z from the figure.



Solution: The general form is $x(t) = \text{Re}[Ce^{zt}]$. Write $C = |C|e^{i\theta}$ and $z = \alpha + i\omega$:

$$x(t) = |C|e^{\alpha t} \cos(\omega t + \theta)$$

Reading from the figure:

- **Initial value:** $x(0) \approx 2$ and the motion starts at a maximum (cosine-like), so $\theta = 0$ and $|C| = 2$. Therefore $C = 2$.
- **Period:** Successive peaks occur near $t \approx 0, 4, 8$, giving $T \approx 4$. Thus

$$\omega = \frac{2\pi}{T} \approx \frac{2\pi}{4} = \frac{\pi}{2} \approx 1.5$$

- **Decay rate:** The exponential envelope (red dashed) decreases from 2 at $t = 0$ to about 1.3 near $t \approx 4$. From $2e^{4\alpha} \approx 1.3$:

$$\alpha \approx \frac{\ln(1.3/2)}{4} \approx \frac{-0.43}{4} \approx -0.1$$

$$\boxed{C \approx 2, \quad z \approx -0.1 + 1.5i}$$

Check: $\text{Re}[2e^{(-0.1+1.5i)t}] = 2e^{-0.1t} \cos(1.5t)$. At $t = 0$: $x = 2 \checkmark$. Period = $2\pi/1.5 \approx 4.2 \checkmark$.

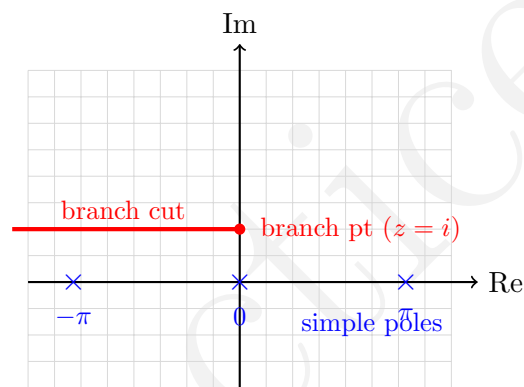
- (10) 5. In the complex plane, identify all singularities and branch points for the following function. If there are branch points, sketch an allowed choice of branch cut.

$$f(z) = \frac{\sqrt{z-i}}{\sin z}$$

Solution:

Branch point: The square root $\sqrt{z-i}$ has a **branch point at** $z = i$. An allowed branch cut runs from $z = i$ to infinity (e.g. upward along the imaginary axis).

Poles: $\sin z = 0$ at $z = n\pi$ for all integers n . Since $\sqrt{n\pi - i} \neq 0$ and $\frac{d}{dz} \sin z|_{n\pi} = \cos(n\pi) = \pm 1 \neq 0$, these are all **simple poles**.



(10) 6. Which, if any, of the following functions are analytic as a function of $z = x + iy$ either everywhere, or with at most isolated singularities?

- A. $f_1 = x^2 - y^2 + 2ixy - 1$
- B. $f_2 = |z|^2$
- C. $f_3 = z^*/|z|^2$
- D. $f_4 = \frac{1}{3}e^{2x}(\cos 2y + i \sin 2y)$

Solution:

A. Note that $(x + iy)^2 = x^2 - y^2 + 2ixy$. So $f_1 = z^2 - 1$.

$\Rightarrow$ **Analytic** (polynomial in z). ✓

B. $f_2 = |z|^2 = z\bar{z}$. This depends on $\bar{z}$, not just z . Check Cauchy–Riemann: $u = x^2 + y^2$, $v = 0$, so $\partial_x u = 2x \neq \partial_y v = 0$ (except at origin).

$\Rightarrow$ **Not analytic**.

C. $f_3 = \bar{z}/|z|^2 = \bar{z}/(z\bar{z}) = 1/z$.

$\Rightarrow$ **Analytic** (except for an isolated singularity at $z = 0$). ✓

D. $f_4 = \frac{1}{3}e^{2x}(\cos 2y + i \sin 2y) = \frac{1}{3}e^{2x} e^{2iy} = \frac{1}{3}e^{2(x+iy)} = \frac{1}{3}e^{2z}$.

$\Rightarrow$ **Analytic** (entire function). ✓

7. Suppose $f(z)$ is a complex analytic function on the complex plane. Let $f(z) = u(x, y) + iv(x, y)$ where u and v are the real and imaginary parts of f , and x, y are the real and imaginary parts of z .

(5) (a) Show that u and v satisfy the 2-D Laplace equation:

$$\nabla^2 u = (\partial_x^2 + \partial_y^2)u = 0$$

$$\nabla^2 v = (\partial_x^2 + \partial_y^2)v = 0$$

Solution: Since $f(z)$ is analytic, the **Cauchy–Riemann conditions** hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

For u : differentiate the first CR equation by x and the second by y :

$$\partial_x^2 u = \partial_x \partial_y v, \quad \partial_y^2 u = -\partial_y \partial_x v$$

Since mixed partials commute ($\partial_x \partial_y v = \partial_y \partial_x v$):

$$\nabla^2 u = \partial_x^2 u + \partial_y^2 u = \partial_x \partial_y v - \partial_y \partial_x v = 0 \quad \checkmark$$

Similarly for v : differentiate the second CR by x and the first by y :

$$\partial_x^2 v = -\partial_x \partial_y u, \quad \partial_y^2 v = \partial_y \partial_x u$$

$$\nabla^2 v = \partial_x^2 v + \partial_y^2 v = -\partial_x \partial_y u + \partial_y \partial_x u = 0 \quad \checkmark$$

(5) (b) Show that the contours of u are perpendicular to the contours of v .

Solution: The gradient ∇u is perpendicular to contours of constant u . Similarly $\nabla v \perp$ contours of constant v .

Contours of u are $\perp$ contours of v if and only if $\nabla u \cdot \nabla v = 0$:

$$\begin{aligned} \nabla u \cdot \nabla v &= (\partial_x u)(\partial_x v) + (\partial_y u)(\partial_y v) \\ &\stackrel{\text{CR}}{=} (\partial_y v)(\partial_x v) + (-\partial_x v)(\partial_y v) \\ &= 0 \quad \checkmark \end{aligned}$$

Since $\nabla u \perp \nabla v$, their respective contour lines are perpendicular at every point of intersection.

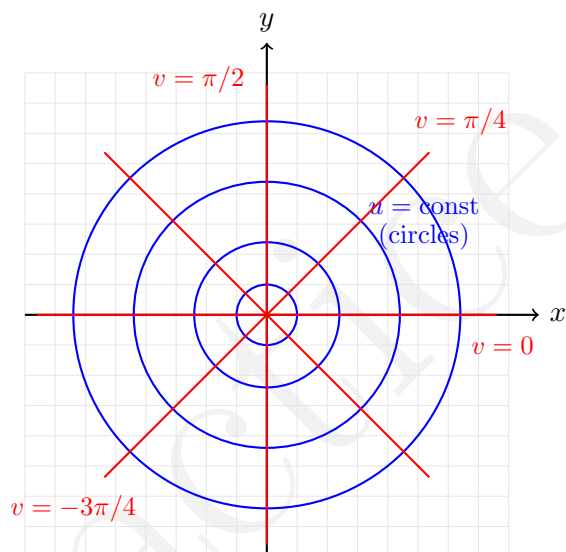
(10) (c) Sketch the contours of u and v for $f(z) = \ln z$.

Solution: Write $z = r e^{i\theta}$:

$$f(z) = \ln z = \underbrace{\ln r}_u + i \underbrace{\theta}_v$$

- Contours of $u = \ln r = \text{const} \Leftrightarrow r = \text{const} \Rightarrow$ **circles centered at the origin.**
- Contours of $v = \theta = \text{const} \Rightarrow$ **rays from the origin.**

The circles and rays are everywhere perpendicular, consistent with part (b).



- (5) 8. (a) Evaluate $\epsilon^{ijk}\epsilon_{ijk}$ where ϵ is the 3D Levi-Civita symbol and we use the Einstein summation convention.

Solution: We sum over all $i, j, k \in \{1, 2, 3\}$:

$$\epsilon^{ijk}\epsilon_{ijk} = \sum_{i,j,k} (\epsilon^{ijk})^2$$

$\epsilon^{ijk} = 0$ unless i, j, k are all different. The non-zero terms correspond to permutations of $(1, 2, 3)$. For example:

$$\epsilon^{123}\epsilon_{123} + \epsilon^{132}\epsilon_{132} + \epsilon^{213}\epsilon_{213} + \cdots = (+1)^2 + (-1)^2 + (-1)^2 + \cdots$$

There are $3! = 6$ permutations of three objects, and each contributes $(\pm 1)^2 = +1$.

$\epsilon^{ijk}\epsilon_{ijk} = 6 = 3!$

- (5) (b) In n dimensions, the Levi-Civita symbol has n indices, each of which takes values $1 \cdots n$, and is defined by the property that it is totally antisymmetric (that is, that it flips sign if any pair of indices is swapped) and that $\epsilon^{12\cdots n} = +1$. What is the value of $\epsilon^{i_1 i_2 \cdots i_n} \epsilon_{i_1 i_2 \cdots i_n}$, where all n of the indices are paired and summed?

Solution: By the same argument: the only non-zero terms have all n indices distinct, i.e. they form permutations of $(1, 2, \dots, n)$. Each such term contributes $(\pm 1)^2 = +1$.

The number of permutations of n objects is $n!$.

$\epsilon^{i_1 \cdots i_n} \epsilon_{i_1 \cdots i_n} = n!$