

Midterm 1

Time Limit: 75 minutes

Name (Print): SOLUTIONS

Student ID: _____

- Do not turn this page until told to do so.
- Hand your exam in before leaving the room.
- Please write your name on every page and your student ID on the first page.
- This exam contains 10 pages (including this cover page) and 6 problems.
- This is a closed book exam. **No books, notes, calculators, or phones allowed.**
- Write all your work on these sheets. If you need more space, you may use the back of the front page; clearly indicate when you have done this.
- **Organize your work**, in a reasonably neat and coherent way.
- **Mysterious or unsupported answers will not receive full credit.**

Problem	Points	Score
1	20	
2	15	
3	15	
4	20	
5	15	
6	15	
Total:	100	

Problem 1. Taylor series.

1.a. Write the first 3 nonzero terms in the Taylor series about the origin for $\frac{1}{1-x^2}$. **[5 pts]**

This is a geometric series with $u = x^2$:

$$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = 1 + x^2 + x^4 + \cdots$$

1.b. Use a Taylor expansion to evaluate **[5 pts]**

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}.$$

Expand $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$, so

$$\frac{e^x - 1 - x}{x^2} = \frac{\frac{x^2}{2} + \frac{x^3}{6} + \cdots}{x^2} = \frac{1}{2} + \frac{x}{6} + \cdots \xrightarrow{x \rightarrow 0} \boxed{\frac{1}{2}}$$

1.c. Write the first 3 nonzero terms in the Taylor series about the origin for $\ln(1+x)$. [5 pts]

Method 1 (geometric series): $\frac{d}{dx} \ln(1+x) = \frac{1}{1+x} = 1 - x + x^2 - \dots$. Integrate term by term:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Method 2 (Taylor series formula): Let $f(x) = \ln(1+x)$. Compute derivatives at $x=0$:

$$\begin{array}{ll} f(x) = \ln(1+x) & f(0) = 0 \\ f'(x) = \frac{1}{1+x} = (1+x)^{-1} & f'(0) = 1 \\ f''(x) = -(1+x)^{-2} & f''(0) = -2 \\ f'''(x) = 2(1+x)^{-3} & f'''(0) = -6 \end{array}$$

The Taylor series $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$ gives:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

1.d. Evaluate the sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

[5 pts]

From part (c), $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

Setting $x=1$:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln(1+1) = \boxed{\ln 2}$$

Problem 2.**2.a.** Express $(1+i)^6$ in polar form $re^{i\theta}$.**[5 pts]**

Convert to polar: $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$ and $\theta = \arctan(1/1) = \pi/4$ (first quadrant), so $1+i = \sqrt{2}e^{i\pi/4}$.

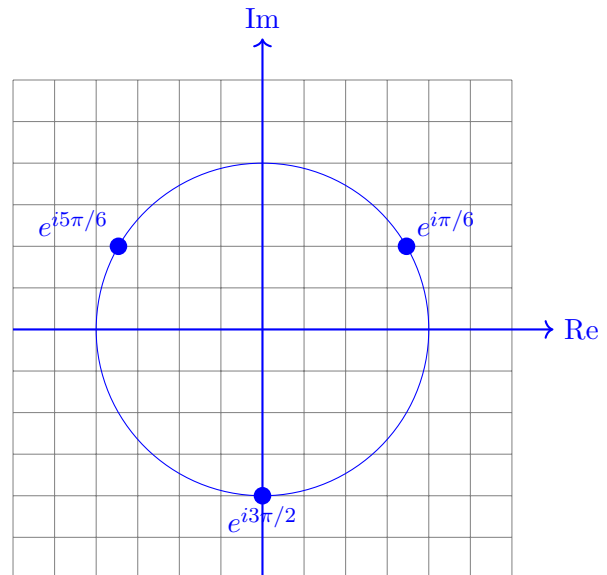
Then:

$$(1+i)^6 = (\sqrt{2})^6 e^{i \cdot 6\pi/4} = 8e^{i \cdot 3\pi/2} = \boxed{8e^{i3\pi/2}}$$

(This equals $-8i$ in Cartesian form.)**2.b.** Find all cube roots of i , i.e., find all complex numbers z such that $z^3 = i$. Express each root in polar form $re^{i\theta}$ and sketch them in the complex plane.**[10 pts]**

Write $i = e^{i\pi/2}$. Then $z^3 = e^{i(\pi/2+2\pi k)}$, so $z = e^{i(\pi/6+2\pi k/3)}$ for $k = 0, 1, 2$:

$$\boxed{z_0 = e^{i\pi/6}, \quad z_1 = e^{i5\pi/6}, \quad z_2 = e^{i3\pi/2}}$$

All have $r = 1$. They are equally spaced at 120 apart on the unit circle.

Problem 3. You may use the fact that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

3.a. Evaluate $\int_{-\infty}^{\infty} e^{-a^2 x^2} dx$. [5 pts]

Substitute $u = ax$, so $du = a dx$:

$$\int_{-\infty}^{\infty} e^{-a^2 x^2} dx = \frac{1}{a} \int_{-\infty}^{\infty} e^{-u^2} du = \boxed{\frac{\sqrt{\pi}}{a}}$$

3.b. Evaluate $\int_{-\infty}^{\infty} x^4 e^{-x^2} dx$. [5 pts]

Consider $I(a) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi} a^{-1/2}$. Differentiating under the integral sign:

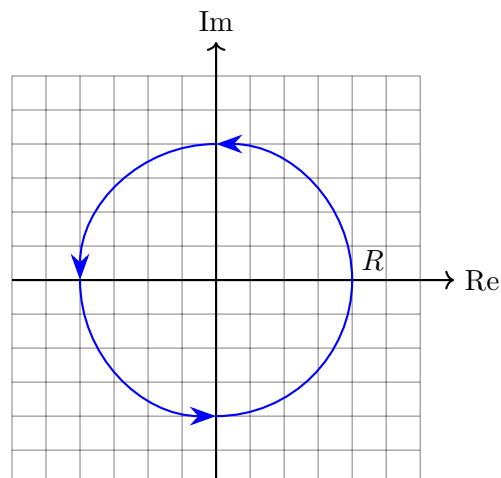
$$\frac{d}{da} I = \int_{-\infty}^{\infty} (-x^2) e^{-ax^2} dx = -\frac{1}{2} \sqrt{\pi} a^{-3/2}$$

$$\frac{d^2}{da^2} I = \int_{-\infty}^{\infty} x^4 e^{-ax^2} dx = \frac{3}{4} \sqrt{\pi} a^{-5/2}$$

Setting $a = 1$:

$$\int_{-\infty}^{\infty} x^4 e^{-x^2} dx = \boxed{\frac{3\sqrt{\pi}}{4}}$$

3.c. Evaluate $\oint_C \bar{z} dz$, where $\bar{z}$ denotes the complex conjugate of z and C is the contour shown below. [5 pts]



Method 1 (parametrize): $z = Re^{i\theta}$, so $\bar{z} = Re^{-i\theta}$ and $dz = iRe^{i\theta} d\theta$.

$$\oint_C \bar{z} dz = \int_0^{2\pi} Re^{-i\theta} \cdot iRe^{i\theta} d\theta = iR^2 \int_0^{2\pi} d\theta = \boxed{2\pi i R^2}$$

Method 2 (residue theorem): On the circle $|z| = R$, we have $z\bar{z} = |z|^2 = R^2$, so $\bar{z} = R^2/z$. Therefore:

$$\oint_C \bar{z} dz = R^2 \oint_C \frac{dz}{z}$$

The integrand $1/z$ is analytic except for a simple pole at $z = 0$ with residue 1. By the residue theorem:

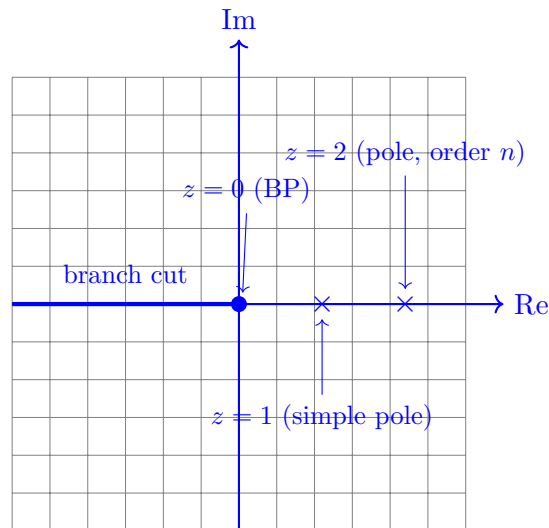
$$R^2 \oint_C \frac{dz}{z} = R^2 \cdot 2\pi i \cdot 1 = \boxed{2\pi i R^2}$$

Problem 4.**4.a.** Consider the function**[5 pts]**

$$g(z) = \frac{\sqrt{z}}{(z-1)(z-2)^n}.$$

Find and classify all singularities and branch points. Sketch the complex plane and indicate the locations of the singularities and branch cuts.

- $z = 0$: **branch point** (from $\sqrt{z}$)
- $z = 1$: **simple pole**
- $z = 2$: **pole of order n**
- Branch cut from $z = 0$ along the negative real axis (standard choice).

**4.b.** Let $u(x, y) = x^2 - y^2$. Find $v(x, y)$ such that $f(z) = u + iv$ is analytic.**[5 pts]**

Method 1 (Cauchy–Riemann): The CR conditions give $v_y = u_x = 2x$ and $v_x = -u_y = 2y$.
From $v_y = 2x$: integrate with respect to y :

$$v = 2xy + g(x)$$

From $v_x = 2y$: $v_x = 2y + g'(x) = 2y$, so $g'(x) = 0$, hence $g(x) = C$.

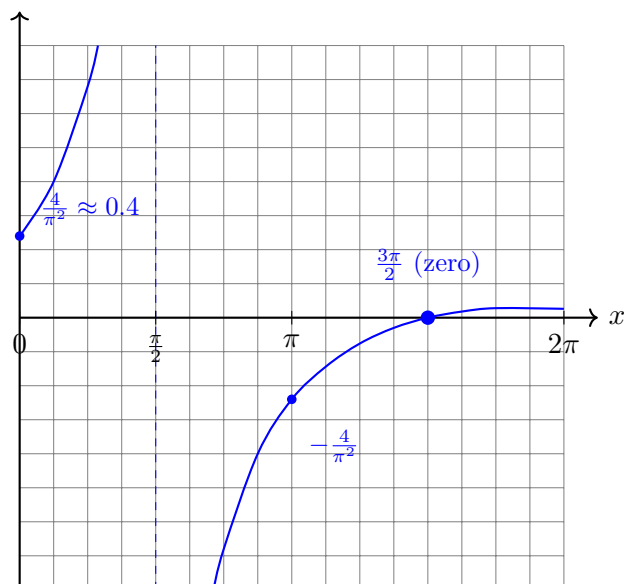
$$\boxed{v(x, y) = 2xy + C}$$

Method 2 (guess the analytic function): We need $f(z)$ analytic with $\operatorname{Re} f = x^2 - y^2$. Since f is analytic, it must be a function of z only (not $\bar{z}$). The real part is quadratic in x, y , so f must be quadratic in z . The only such function (up to a constant) is z^2 :

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

which has $\operatorname{Re} z^2 = x^2 - y^2 = u$ ✓. Therefore $v = \operatorname{Im} z^2 = \boxed{2xy + C}$.

4.c. Consider the real function $f(x) = \frac{\cos x}{(x - \pi/2)^2}$. Sketch $f(x)$ for $0 \leq x \leq 2\pi$. Indicate all zeros and singularities on your sketch. [5 pts]



Key features:

- Vertical asymptote at $x = \pi/2$: $f \rightarrow +\infty$ from the left, $f \rightarrow -\infty$ from the right (since $\cos x \approx -(x - \pi/2)$ near $\pi/2$).
- Zero at $x = 3\pi/2$ (where $\cos x = 0$ but denominator is finite).
- $f(0) = 4/\pi^2 \approx 0.4$, $f(\pi) = -4/\pi^2 \approx -0.4$, $f(2\pi) = 4/(9\pi^2) \approx 0.045$.

4.d. Now think of $f(z) = \frac{\cos z}{(z - \pi/2)^2}$ as a function of the complex variable z . Classify the singularity at $z = \pi/2$. [5 pts]

Let $w = z - \pi/2$. Then $\cos z = \cos(\pi/2 + w) = -\sin w = -w + w^3/6 - \dots$

$$\frac{\cos z}{(z - \pi/2)^2} = \frac{-w + w^3/6 - \dots}{w^2} = \frac{-1}{w} + \frac{w}{6} - \dots$$

The Laurent series has a $1/w$ term as the leading singularity. Therefore $z = \pi/2$ is a simple pole.
(Despite the $(z - \pi/2)^2$ in the denominator, the zero of $\cos z$ at $\pi/2$ cancels one power.)

Problem 5. Use contour integration to evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)(x^2 + 4)}.$$

5.a. Consider $f(z) = \frac{1}{(z^2 + 1)(z^2 + 4)}$. Find all the poles of $f(z)$ in the **upper half-plane**. [5 pts]

Factor: $z^2 + 1 = (z - i)(z + i)$ and $z^2 + 4 = (z - 2i)(z + 2i)$.

Poles: $z = \pm i$ and $z = \pm 2i$. In the upper half-plane:

$$z = i \quad \text{and} \quad z = 2i$$

5.b. Compute the residue of $f(z)$ at each pole you found in part (a). [5 pts]

At $z = i$ (simple pole):

$$\text{Res}_{z=i} = \left. \frac{1}{(z+i)(z^2+4)} \right|_{z=i} = \frac{1}{(2i)(i^2+4)} = \frac{1}{(2i)(3)} = \frac{1}{6i} = \boxed{-\frac{i}{6}}$$

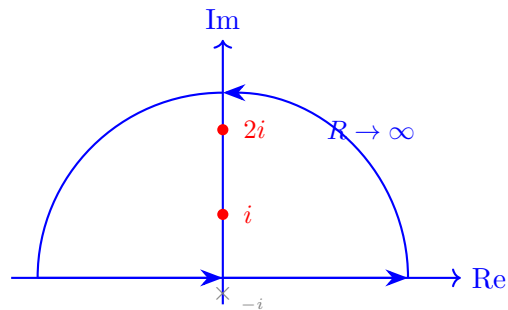
At $z = 2i$ (simple pole):

$$\text{Res}_{z=2i} = \left. \frac{1}{(z^2+1)(z+2i)} \right|_{z=2i} = \frac{1}{((2i)^2+1)(4i)} = \frac{1}{(-3)(4i)} = \frac{-1}{12i} = \boxed{\frac{i}{12}}$$

5.c. Close the contour in the upper half-plane with a large semicircular arc. You may assume the integral over the arc vanishes as the radius goes to infinity. Use the residue theorem to evaluate I . [5 pts]

By the residue theorem:

$$I = 2\pi i \sum \text{Res} = 2\pi i \left(-\frac{i}{6} + \frac{i}{12} \right) = \boxed{\frac{\pi}{6}}$$



Problem 6.**6.a.** Evaluate $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B})$. **[5 pts]** $\mathbf{A} \times \mathbf{B}$ is perpendicular to $\mathbf{A}$, so $\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = \boxed{0}$.Alternatively, in index notation: $A_i \epsilon_{ijk} A_j B_k = 0$ because $A_i A_j$ is symmetric in i, j while ϵ_{ijk} is antisymmetric.**6.b.** Using the Einstein summation convention, evaluate: **[5 pts]**

$$\delta_{kk} = \delta_{11} + \delta_{22} + \delta_{33} = \boxed{3}$$

$$\epsilon_{ijk} \delta_{ij} = \epsilon_{iik} = \boxed{0} \quad (\text{repeated index on } \epsilon \text{ forces it to vanish by antisymmetry})$$

$$\epsilon_{3ij} \epsilon_{ji3} = \text{Nonzero terms: } (i, j) = (1, 2): \epsilon_{312} \epsilon_{213} = (1)(-1) = -1. \quad (i, j) = (2, 1): \epsilon_{321} \epsilon_{123} = (-1)(1) = -1. \quad \text{Sum} = \boxed{-2}$$

6.c. Let $\mathbf{A} = (1, 1, 0)$ and $\mathbf{B} = (0, 1, 1)$. Compute the components of $\mathbf{A} \times \mathbf{B}$ and verify by explicit computation that it is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. **[5 pts]**

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \hat{x}(1 \cdot 1 - 0 \cdot 1) - \hat{y}(1 \cdot 1 - 0 \cdot 0) + \hat{z}(1 \cdot 1 - 1 \cdot 0) = \boxed{(1, -1, 1)}$$

Check perpendicularity:

$$\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = (1)(1) + (1)(-1) + (0)(1) = 0 \checkmark$$

$$\mathbf{B} \cdot (\mathbf{A} \times \mathbf{B}) = (0)(1) + (1)(-1) + (1)(1) = 0 \checkmark$$