

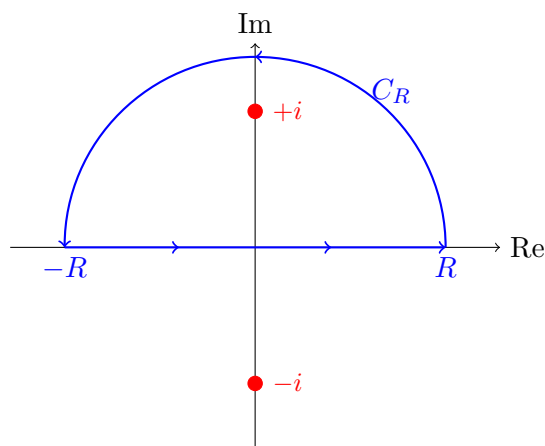
Example: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

We already know the answer is π (from arctan), but let's re-derive it using contour integration.

Step 1: Find the poles

We need the roots of $z^2 + 1 = 0$, i.e. $z^2 = -1$:

$$z = +i \quad (\text{upper half-plane}), \quad z = -i \quad (\text{lower half-plane})$$



Step 2: Choose the contour and check the arc

Close the real-line integral with a semicircle C_R of radius R in the **upper half-plane**.

Parametrize $z = Re^{i\theta}$, $dz = iRe^{i\theta} d\theta$, with θ from 0 to π :

$$\int_{C_R} \frac{dz}{1+z^2} = \int_0^\pi \frac{iRe^{i\theta} d\theta}{1+R^2e^{2i\theta}} \approx \int_0^\pi \frac{iRe^{i\theta} d\theta}{R^2e^{2i\theta}} = \frac{i}{R} \int_0^\pi e^{-i\theta} d\theta = \frac{2}{R} \xrightarrow{R \rightarrow \infty} 0.$$

In the second step we used $|R^2e^{2i\theta}| = R^2 \gg 1$ for all θ , so the 1 in the denominator is negligible.

Shortcut (ML inequality): $|z^2 + 1| \geq R^2 - 1$ on C_R , so

$$\left| \int_{C_R} \frac{dz}{1+z^2} \right| \leq \frac{\pi R}{R^2 - 1} \xrightarrow{R \rightarrow \infty} 0.$$

The arc integral vanishes. One pole is enclosed: $z = +i$.

Step 3: Compute the residue

Factor the denominator: $z^2 + 1 = (z - i)(z + i)$. So

$$f(z) = \frac{1}{(z - i)(z + i)}$$

has a simple pole at $z = i$ with residue:

$$\text{Res}(f, i) = \lim_{z \rightarrow i} (z - i) \frac{1}{(z - i)(z + i)} = \frac{1}{i + i} = \frac{1}{2i}$$

Aside: Residues for higher-order poles

In this example we had a **simple pole** (order 1), and the residue was easy to compute by multiplying by $(z - z_0)$ and taking the limit. What if the pole has higher order?

If $f(z)$ has a pole of order n at $z = z_0$, then near z_0 :

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{(z - z_0)} + a_0 + \cdots$$

The residue is a_{-1} (the coefficient of the $1/(z - z_0)$ term). To extract it, multiply by $(z - z_0)^n$ and differentiate $n - 1$ times:

$$\boxed{\text{Res}(f, z_0) = \frac{1}{(n - 1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)]}$$

Example: $f(z) = \frac{1}{(z - i)^2(z + i)^2} = \frac{1}{(1 + z^2)^2}$ has a double pole at $z = i$.

$$\text{Res}(f, i) = \lim_{z \rightarrow i} \frac{d}{dz} \left[\frac{1}{(z + i)^2} \right] = \lim_{z \rightarrow i} \frac{-2}{(z + i)^3} = \frac{-2}{(2i)^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

This would let us evaluate $\int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2} = 2\pi i \times \frac{1}{4i} = \frac{\pi}{2}$.

Step 4: Apply the residue theorem

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = 2\pi i \times \frac{1}{2i} = \pi}$$

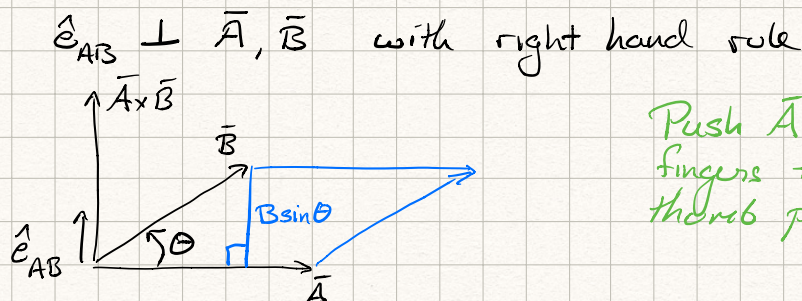
Checks

- **Known result:** $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = [\arctan x]_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$. ✓
- **Sign check:** The integrand is positive, so the result must be positive. ✓
- **Lower half-plane:** We could also close in the lower half-plane (clockwise, picking up $z = -i$). Then $\text{Res}(f, -i) = \frac{1}{-2i}$ and the clockwise orientation gives an extra minus sign: $(-2\pi i) \times \frac{1}{-2i} = \pi$. Same answer. ✓

Vector Calculus

Vector Operations: Summary

- addition $\vec{A} + \vec{B} \rightarrow \text{vector}$
- linear combination $\alpha \vec{A} + \beta \vec{B} \rightarrow \text{vector}$
- dot/inner product $\vec{A} \cdot \vec{B} = AB \cos \theta \rightarrow \text{scalar}$
 - norm/length $A = |\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$
 - in components $\vec{A} \cdot \vec{B} = A^i g_{ij} B^j = A_j B^j \stackrel{\text{ON basis}}{=} \sum_j A^j B^j$
- cross/vector product $\vec{A} \times \vec{B} = AB \sin \theta \hat{e}_{AB} \rightarrow \text{pseudo vector axial vector}$



Push $\vec{A}$ with right hand fingers to $\vec{B}$, thumb points $\hat{e}_{AB}$

- Area of parallelogram: $|\vec{A} \times \vec{B}| = AB |\sin \theta|$

- In components: $(\vec{A} \times \vec{B})^i = \epsilon_{ijk} A^j B^k$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

"free" ϵ_{ijk} sum repeated
violates convenient notation
 $\rightarrow$ only valid in ON basis

$$= \hat{e}_x (A_y B_z - A_z B_y) + \hat{e}_y (A_z B_x - A_x B_z) + \hat{e}_z (A_x B_y - A_y B_x)$$

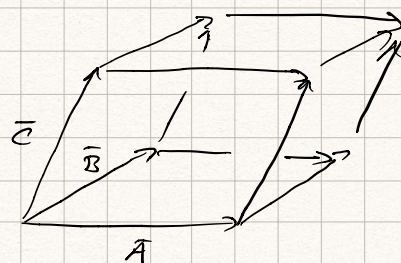
- anticommutative: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Q? Why special to 3d? 2 vectors specify a plane $\rightarrow$ unique $\perp$ vec in 3d

- (scalar) triple product: $\vec{A} \cdot (\vec{B} \times \vec{C}) = \epsilon_{ijk} A^i B^j C^k = \det(\vec{A}, \vec{B}, \vec{C})$

- volume of parallelepiped

- sign +ve if $\vec{A}, \vec{B}, \vec{C}$ right handed



Fields

A **field** is a physical quantity which varies across different points in space and time.

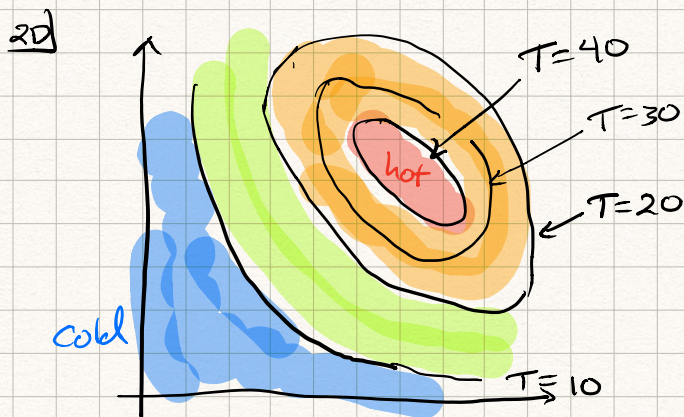
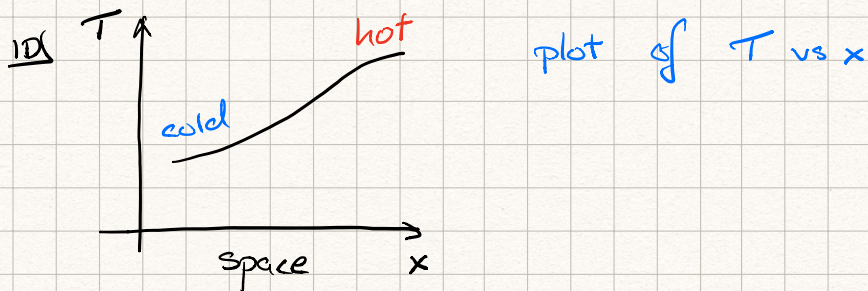
Ex **Scalar fields** target space = $\mathbb{R}$

density (mass, charge, etc) $\rho(\vec{r}, t)$

temperature $T(\vec{r}, t)$

Electrostatic potential $\phi(\vec{r}, t)$

Visualizing scalar fields



Contours = lines of equal value

"isotherm", "equipotential"

topographic map = contour plot of height of ground

heat map / color density plot
color indicates value

3D/4D/...

Slices on 2D. Projections to 2D.

Animation/movies

Interactive manipulation of 3D object (3D printer or VR)

8 Vector fields - vector $\vec{v}$ at every point, except at special points usually smooth
target space = $\mathbb{R}^d$

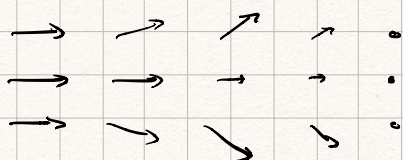
electric field	$\vec{E}(\vec{r}, t)$
magnetic field	$\vec{B}$
velocity of fluid	$\vec{v}$
current density	$\vec{J}(\vec{r}, t) = \rho(\vec{r}, t) \vec{v}(\vec{r}, t)$

Visualization of Vector Fields

1D] Q? How many components in 1D vector?

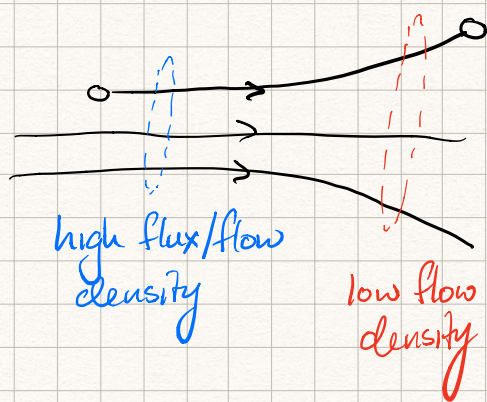


2D]



Arrow/Quiver Plot

Draw sampling of vectors on grid of points



Field Lines/ streamer Lines

Think of flow of water $\vec{J}$

- tangent is direction of vector field
- density indicates magnitude
- field lines start/stop on sources/sinks of flow

Vector Derivatives

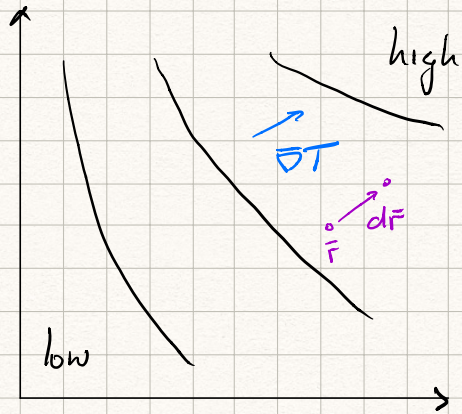
- Fields vary in space $\rightarrow$ calculus to quantify
- All derivatives based on $\vec{\nabla}$ ("del"/"nabla") operators.

Notation: $\hat{x}, \hat{y}, \hat{z} = \hat{e}_x, \hat{e}_y, \hat{e}_z = \hat{e}_1, \hat{e}_2, \hat{e}_3$ Cartesian basis vectors

Points in space represented by radius vector (displacement from 0)

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

1st Derivative of Scalars: the Gradient



$$T(\bar{r} + d\bar{r}) = T(\bar{r}) + dT$$

$\uparrow$ small displacement $\uparrow$ small change in T

← implicitly depends on $\bar{r}, d\bar{r}$
"total differential"

In Cartesian coordinates:

$$T(x+dx, y+dy, z+dz) = T(x, y, z) + \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

$$dT = \underbrace{\left(\hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} \right)}_{\text{scalar}} \cdot \underbrace{d\vec{r}}_{\text{vector}}$$

$$\vec{\nabla} T = \text{grad } T \quad \leftarrow \text{vector field}$$

- ∇T points uphill in direction of steepest ascent
- $|\nabla T|$ rate of change in that direction
- Must be a vector b/c dT is scalar, $d\vec{r}$ is vector so components of ∇T must transform as vectors.

Formally $\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$ is a vector operator

- Operator eats a field and produces another linearly.

$\bar{\partial}$ is hungry! derivatives act to the right

$$f \circ \nabla \neq \nabla \circ f$$

Ex for conservative forces:

$$\bar{F} = -\bar{\nabla} \phi$$

forces $\vec{F}$ point down potential energy ϕ