

Vectors

Geometric Description

vector $\vec{v}$: quantity w/ **magnitude** $|\vec{v}|=v$ and **direction**
(eg. displacement $\vec{s}$, velocity $\vec{v}$, force $\vec{F}$, ...)

scalar k is a quantity with **magnitude** (and sign)
(eg. charge q , temperature T)

Basic Operations

vector addition:

commutative
associative

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

$$(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$$

scalar multiplication:

magnitude

same direction $k > 0$, opposite $k < 0$

distributive

$$k\vec{v}$$

$$|k\vec{v}| = |k||\vec{v}|$$

$$a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$$

$$(a+b)\vec{v} = a\vec{v} + b\vec{v}$$

vector subtraction:

$$\vec{v} - \vec{w} = \vec{v} + (-1 \cdot \vec{w})$$

zero vector:

$$\vec{v} + \vec{0} = \vec{v}, \quad \vec{v} - \vec{v} = \vec{0}$$

Algebraic Description

Introduce **orthonormal basis**:

$\hat{e}_x, \hat{e}_y, \hat{e}_z$

orthogonal

unit length

$$|\hat{e}_i| = 1$$

$\{\hat{e}_i\}$ forms a basis for the vector space



Any $\vec{v}$ is a unique **linear combination** of $\hat{e}_i$:

$$\vec{v} = v^x \hat{e}_x + v^y \hat{e}_y + v^z \hat{e}_z$$

components of $\vec{v}$ in $\hat{e}_i$ basis

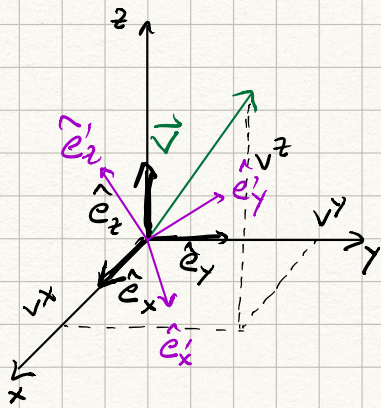
Vector addition/scalar multiplication by components:

$$\vec{u} = a\vec{v} + b\vec{w} \iff u^x = av^x + bw^x, \dots$$

Index notation: $u^i = av^i + bw^i$

$\forall i = x, y, z$

Q? How many equations is this? $\uparrow$ 3 for 3d vectors, d in genl



Can choose any orthogonal unit vectors as basis.
Different bases are related by rotations and reflections.

vector $\vec{v}$ does not depend on basis, but components do.

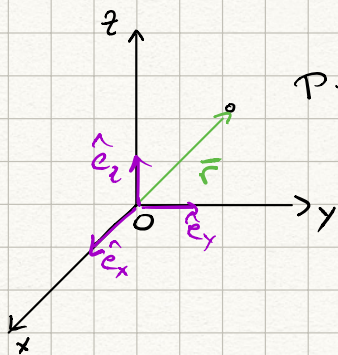
Often write components as column vector

$$\vec{v} = \sum_i v_i \hat{e}_i \longleftrightarrow \begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix}$$

NOT an equality: $\vec{v} \neq \begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix}$ b/c RHS depends on basis choice.

though sometimes people write equality so long as basis is fixed.

Q? Is a point P a vector?



$$P = (x, y, z)$$

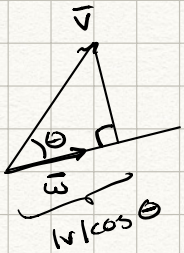
No! But displacement is:

vector from origin to P is radius vector

$$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

Scalar/Dot Product

(vector $\cdot$ vector $\rightarrow$ scalar)



$$\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \Theta_{vw} \quad \text{geometric, indep of coords}$$

↑↑↑ vector ↑↑↑ scalar

Project $\vec{v}$ onto direction of $\vec{w}$, scale by $|\vec{w}|$

commutative:

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

distributive:

$$\vec{v} \cdot (\vec{w} + \vec{u}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{u}$$

Norm/length squared: $\vec{v} \cdot \vec{v} = |\vec{v}|^2$

Schwarz Inequality: $|\vec{v} \cdot \vec{w}| \leq |\vec{v}| |\vec{w}|$

Orthogonality: $\vec{v} \cdot \vec{w} = 0 \Leftrightarrow \vec{v} \perp \vec{w}$ orthogonal

Orthonormal basis:

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases} \quad \text{unit orthogonal}$$

↑
Kronecker Delta symbol

Orthogonality Trick: Dot product w/ ON basis gives components!

$$\hat{e}_x \cdot \vec{v} = \hat{e}_x \cdot (v^x \hat{e}_x + v^y \hat{e}_y + v^z \hat{e}_z) = v^x$$

$$\boxed{\hat{e}_i \cdot \vec{v} = v^i}$$

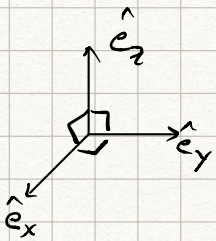
Dot product in components:

$$\vec{v} \cdot \vec{w} = \vec{v} \cdot \left(\sum_i \hat{e}_i w^i \right) = \sum_i \vec{v} \cdot \hat{e}_i w^i$$

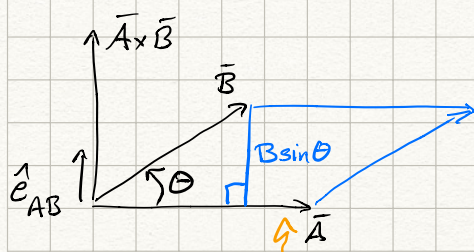
$$\boxed{\vec{v} \cdot \vec{w} = \sum_i v^i w^i \equiv v^i w^i}$$

↑
Einstein summation convention
 $\rightarrow$ sum repeated indices

Pythagorean Thm: $|\vec{v}|^2 = \vec{v} \cdot \vec{v} = v^i v^i = v^x^2 + v^y^2 + v^z^2$



Cross/Vector Product (vector $\times$ vector $\rightarrow$ (pseudo) vector)



$$\vec{A} \times \vec{B} = \underbrace{AB \sin \theta}_{\text{magnitude}} \underbrace{\hat{e}_{AB}}_{\text{direction}}$$

$\hat{e}_{AB} \perp \vec{A}, \vec{B}$ with right hand rule

Push $\vec{A}$ with right hand fingers to $\vec{B}$, thumb points $\hat{e}_{AB}$

• Area of parallelogram: $|\vec{A} \times \vec{B}| = AB |\sin \theta|$

• anticommutative: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (why? right points opp. dir)

Q? Why special to 3d? 2 vectors specify a plane $\rightarrow$ unique $\perp$ vector

• In components:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (\text{will return to determinant later})$$

$$= \hat{e}_x (A_y B_z - A_z B_y) + \hat{e}_y (A_z B_x - A_x B_z) + \hat{e}_z (A_x B_y - A_y B_x)$$

• Index notation: $(\vec{A} \times \vec{B})^i = \underbrace{\epsilon^{ijk}}_{\text{free}} \overbrace{A^j B^k}^{\text{sum on } j, k}$

def The Levi-Civita symbol ϵ^{ijk} is the completely antisymmetric symbol with 3 indices such that $\epsilon^{xyz} = +1$.

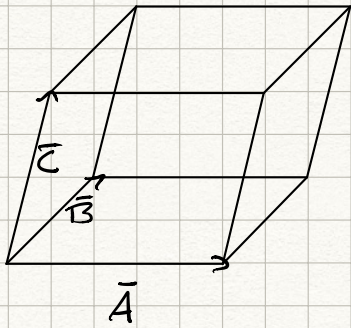
complete antisymmetry: swap any 2 indices, get a - sign

$$\epsilon^{ijk} = -\epsilon^{jik}$$

$$\epsilon^{ijk} = -\epsilon^{kji}$$

Equivalently, $\epsilon^{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of } xyz \\ -1 & \text{if " " odd " "} \\ 0 & \text{if any indices agree} \end{cases}$

(Scalar) triple product



$$\begin{aligned}\bar{A} \cdot (\bar{B} \times \bar{C}) &= A^i (\bar{B} \times \bar{C})^i = A^i \epsilon^{jk} B^j C^k \\ &= \epsilon^{ijk} A^i B^j C^k\end{aligned}$$

- oriented volume of // - piped
- sign (orientation) +ve if A, B, C right handed

Also can be written: $\begin{vmatrix} A^x & A^y & A^z \\ B^x & B^y & B^z \\ C^x & C^y & C^z \end{vmatrix}$ if you know determinants

