

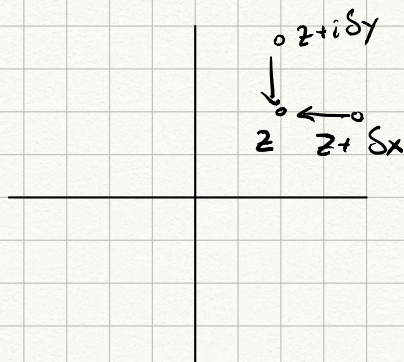
Complex Analytic Functions

cf Shankar Ch 6

The derivative of $f(z)$ wrt z is defined

$$\frac{df}{dz} = \lim_{\delta z \rightarrow 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

where the limit must not depend on direction of δz in $\mathbb{C}$ -plane:



$$\delta z = \delta x + i\delta y$$

This is stronger constraint than for real functions, where limit must agree only from 2 directions.

Consider f as 2D function:

$$f(z) = \text{Re}(f(z)) + i \text{Im}(f(z)) = u(x, y) + i v(x, y)$$

Then:

$$\frac{\delta f}{\delta z} = \frac{\delta u + i \delta v}{\delta x + i \delta y}$$

for $\delta z = \delta x$ along real direction:

$$\frac{\delta f}{\delta z} = \frac{\delta u + i \delta v}{\delta x} \xrightarrow{\delta x \rightarrow 0} \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

partial derivatives

for $\delta z = i\delta y$

|| want

$$\frac{\delta f}{\delta z} = \frac{\delta u + i \delta v}{i \delta y} \xrightarrow{\delta y \rightarrow 0} -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

If $\frac{df}{dz}$ is defined, these must agree:

$\therefore$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Cauchy-Riemann
Conditions

Props

① $\frac{df}{dz}$ exists $\iff$ Cauchy-Riemann holds

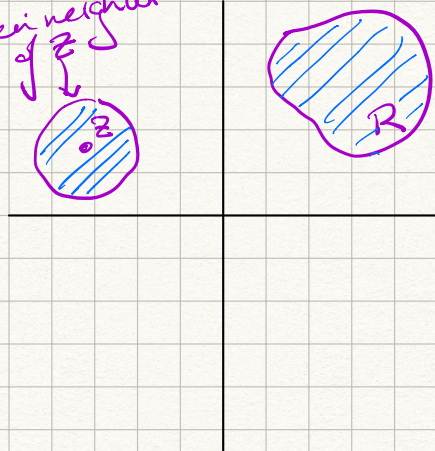
② If $\frac{df}{dz}$ exists in some open region R of $\mathbb{C}$, say

f is analytic in R

If f is analytic in open neighborhood of z , then analytic at z .

If $R = \mathbb{C}$, say f is entire.

open neighborhood
of z



(open means don't include the boundary
 $\rightarrow$ every point inside has a small neighborhood inside)

Ex z^2 is analytic

pf/ $z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2xy}_{v(x,y)} i$

CR: $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ entire $\checkmark$

Ex z^* is not analytic

$$z^* = (x+iy)^* = \underbrace{x}_{u} + i \underbrace{(-y)}_{v}$$

$$\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$$

③ Analytic functions only depend on z , not z^* :

General function of 2 variables (x, y)

$$f(x, y)$$

can change variables to complex coordinates (z, z^*)

$$z = x + iy$$

$$x = (z + z^*)/2$$

$$z^* = x - iy$$

$$y = (z - z^*)/2i$$

$$\text{eg. } f(x, y) = x^2 - y^2 = (z + z^*)^2/4 + (z - z^*)^2/4 \\ = \frac{1}{2}(z^2 + z^{*2})$$

This depends on z^* , so should write

$$f(z, z^*) = \frac{1}{2}(z^2 + z^{*2}) \quad \text{not analytic!}$$

Analytic functions don't depend on z^* if written in z, z^* coordinates.

$$f(z, z^*) = f(z)$$

④ Computation of $f'(z) = \frac{df}{dz}$:

If $f(z)$ has series representation in z , it satisfies CR and can be differentiated using standard rules.

Ex $\frac{d}{dz}(z^3) = 3z^2 \quad \frac{d}{dz} \sin z = \cos z$

$$\frac{d}{dz}(f(z)g(z)) = \frac{df}{dz}g + f\frac{dg}{dz}$$

Generally, if you can write f using elementary functions of z

(addition, multiplication, roots, exp, etc...) and no z^* 's, can use standard derivative rules.

Singularities of Analytic Function

Most analytic functions are not entire.

Singularities are isolated points where f is not analytic

Simple poles: (Important!)

If $f(z)$ has an isolated singularity at z_0 but

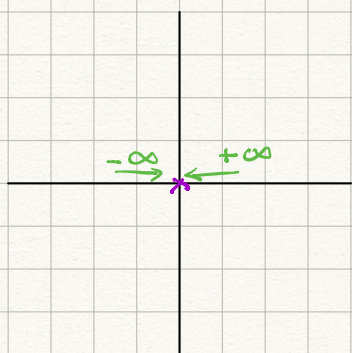
$$R(z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z) \text{ is finite}$$

then f has a simple pole with residue R .

Alternatively:

$$f(z) \rightarrow \frac{R(z_0)}{(z - z_0)} (1 + O(z - z_0))$$

Ex



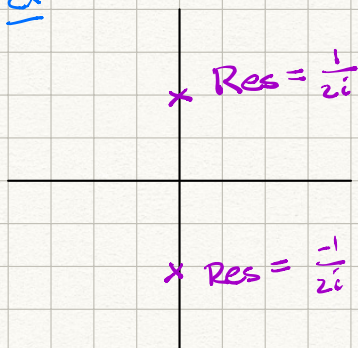
$$f(z) = \frac{1}{z} \quad \leftarrow \text{residue}$$

Analytic except @ $z=0$. (not even continuous)

simple pole @ $z=0$ with residue 1

↓
"Healed" by multiplication by z_0

Ex



$$f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z+i)(z-i)}$$

Singularities @ $\pm i$.

$$\text{Near } z_0 = +i: \frac{1}{(z-i)} \cdot \frac{1}{z+i} = \frac{1}{z-i} \cdot \frac{1}{2i + (z-i)}$$

$$= \frac{1/2i}{z-i} \cdot \frac{1}{1 + \frac{z-i}{2i}} \approx \frac{1/2i}{z-i} (1 + \dots)$$

$$R(z_0) = 1/2i$$

$$\text{Similarly near } -i: \frac{1}{z+i} \cdot \frac{1}{z-i} \rightarrow R = -\frac{1}{2i}$$

Higher order poles

f has n^{th} order pole @ z_0 if $\lim_{z \rightarrow z_0} (z-z_0)^n f(z)$ is finite.

i.e.

$$f(z) \rightarrow \frac{C}{(z-z_0)^n} (1 + O(z-z_0)) \quad \text{near } z=z_0$$

Ex Q?

$$\frac{1}{(z-1)^2}$$

Essential Singularities

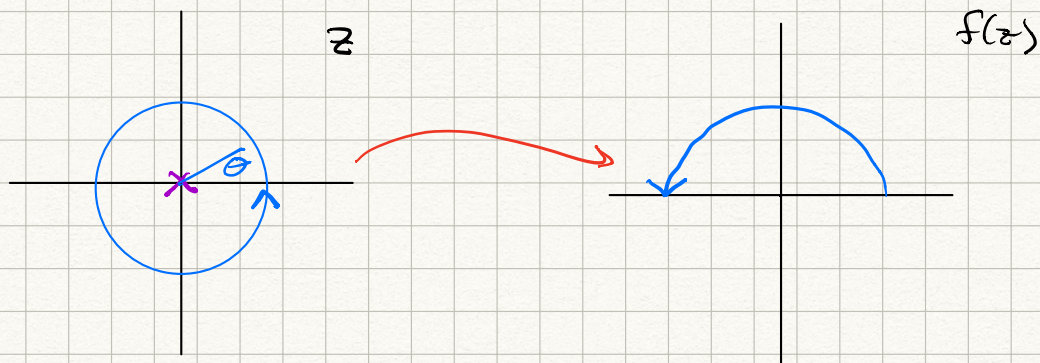
f has essential sing @ z_0 if f has a pole but arbitrarily high powers of n fail to make f finite.

Ex $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$

Branch Points

$f(z)$ has a branch point @ z_0 if, on encircling z_0 , f goes to a different value:

$$f(z) = \sqrt{z} = z^{1/2} = r^{1/2} e^{i\theta/2}$$



fix r , follow $f(re^{i\theta})$ as θ goes from 0 to 2π

$$\theta = 0 \quad \longrightarrow \quad r^{1/2}$$

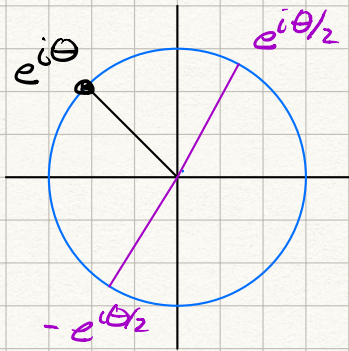
$$\theta = 2\pi \quad \longrightarrow \quad r^{1/2} e^{i2\pi/2} = -r^{1/2}$$

$$\theta = 4\pi \quad \longrightarrow \quad r^{1/2} e^{i4\pi/2} = r^{1/2}$$

What??

Multivalued functions and Branches

Many important functions become "multivalued" when extended to $\mathbb{C}$.



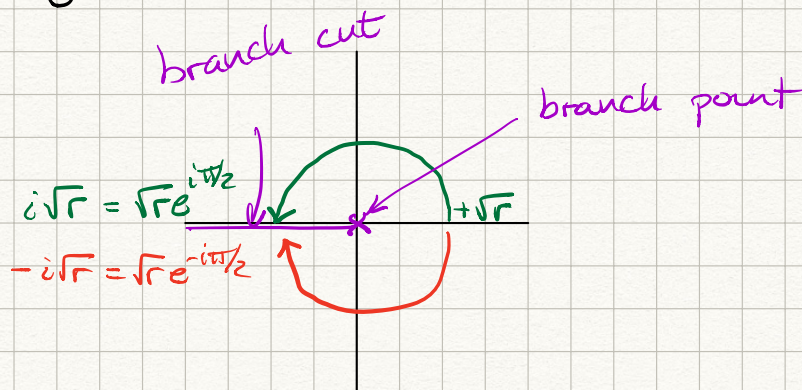
Two square roots of z .

For real numbers $(\pm 3)^2 = 9$ so not a surprise.

But we need to be careful how we choose which "branch" of square roots we mean by $\sqrt{z}$ or $z^{1/2}$.

Typically use "principal branch" defined so that

$\sqrt{z}$ is real and positive for z real and positive, and smooth everywhere except on "branch cut" along negative real axis where $\sqrt{z}$ jumps.

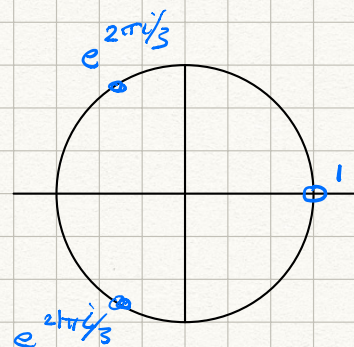


Ex $z^{1/2} = r^{1/4} e^{i(\theta + 2\pi n)/2}$

So z has g distinct g^{th} roots, $n = 0, 1, \dots, g-1$

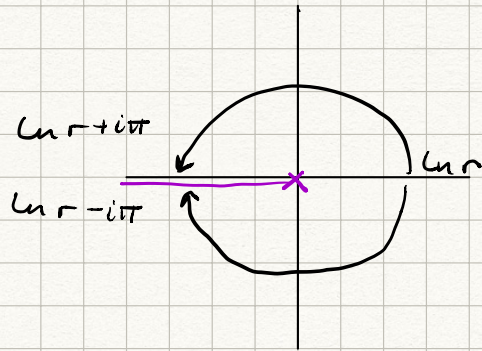
And $z^{1/g}$ has g distinct branches

Q? What are the 3rd roots of unity?



Ex $\ln z = \ln r e^{i(\theta+2\pi n)} = \ln r + i(\theta+2\pi n)$

Logarithm has infinitely many branches b/c $e^{i\theta}$ is periodic



Principal branch