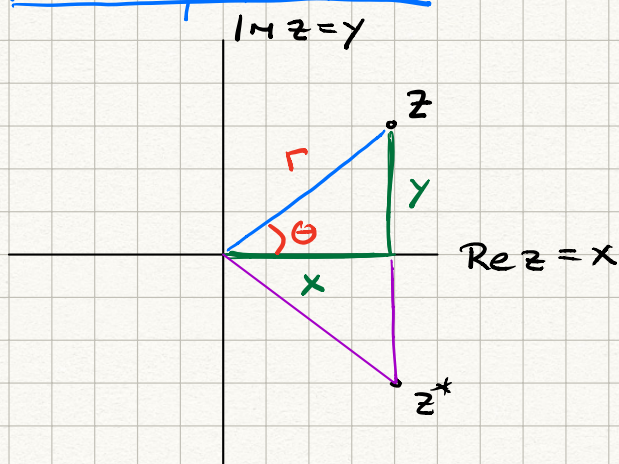


The Complex Plane



Can view $\mathbb{C}$ numbers

$$z = x + iy = r e^{i\theta}$$

as points in 2D plane w/
Cartesian coordinates (x, y) .
or polar coordinates (r, θ) .

Python/NumPy

$$i = \sqrt{-1}$$

$$x = \operatorname{Re} z = \frac{z + z^*}{2}$$

$$y = \operatorname{Im} z = \frac{z - z^*}{2i}$$

$$r = |z| = \sqrt{z z^*}$$

$$\theta = \arg z = \tan^{-1} \frac{y}{x}$$

$$z^* = (x + iy)^* = x - iy$$

"i"

"real part"

"imaginary part"

"norm/modulus/
magnitude"

"argument/phase/
angle"

"complex conjugate"

1j

`real(z)` or `z.real`

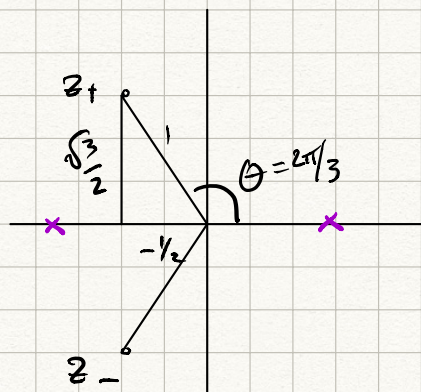
`imag(z)` or `z.imag`

`abs(z)`

`angle(z)`

`conj(z)`

Ex Complex roots of $z^2 + z + 1 = 0$



$$z_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i = e^{\pm i 2\pi/3}$$

$$r^2 = |z_{\pm}|^2 = \frac{1}{4} + \frac{3}{4} = 1$$

$$\theta = \pm 2\pi/3$$

Q? what about $z^2 + z - 1 = 0$
 $z = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ real

Functions of Complex Variables

If a real function has convergent power series

$$f(x) = \sum a_n x^n$$

can extend to a complex function

$$f(z) = \sum a_n z^n$$

which agrees on the real line $z=x$.

Ex The Complex Exponential function:

$$e^z \equiv \sum_n \frac{z^n}{n!}$$

Converges for any z (check later)

Note:
$$e^{i\theta} = \sum_n \frac{(i\theta)^n}{n!}$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

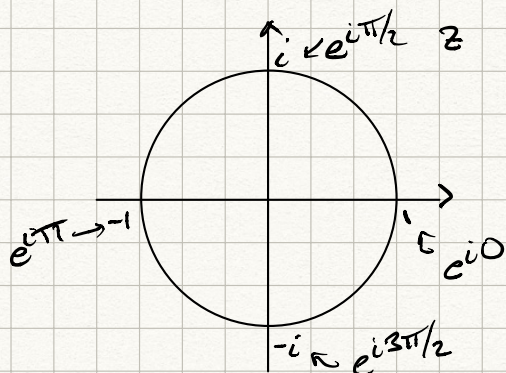
$$e^{i\theta} = \cos \theta + i \sin \theta$$

Euler formula

underlies polar form of z :

$$z = x + iy = r \left[\frac{x}{r} + i \frac{y}{r} \right] = r (\cos \theta + i \sin \theta) = r e^{i\theta}$$

Ex The unit circle $|z| = 1$



$$z = e^{i\theta}$$

Q? $e^{i(3\pi)}$?

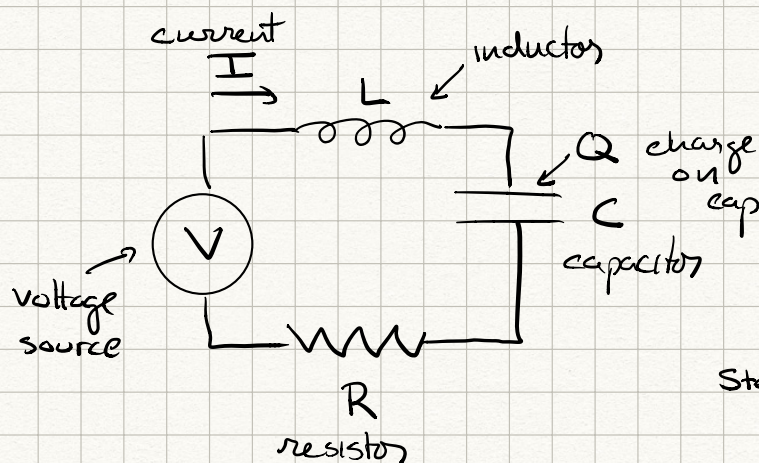
Periodicity: $e^{z+2\pi i} = e^z e^{2\pi i} = e^z$

Application to Linear Circuits

Why introduce $\mathbb{C}$ -numbers in physics?

- quantum mechanics takes place in complex Hilbert space
- conserved particles associated with complex fields
- simplifies analysis of linear systems

Let's expand on this



Goal: Find $I(t)$
given applied $V(t) = V_0 \cos \omega t$

Why $\cos \omega t$? → AC generators
→ any function can be written as
superposition of sinusoidal functions

Step 1: Note $\frac{dQ}{dt} = I$
and work with $Q(t)$.

Kirchhoff's Law:
$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = V(t)$$

→ Linear ordinary differential equation (more later)

single
independent
variable t

relates Q and its derivatives

→ if $Q_1(t)$ solves eqn with $V_1(t)$

and $Q_2(t)$ " " " $V_2(t)$

then $aQ_1(t) + bQ_2(t)$ solves equation for $aV_1(t) + bV_2(t)$.

Solve by "ansatz" = "guess with parameters".

Warm up with only inductor L :

$$L \ddot{Q} = V_0 \cos \omega t$$

Need function whose 2nd derivative is cosine: Cosine!

Ansatz: $Q_0 \cos \omega t$

Plug in: $-L Q_0 \omega^2 \cancel{\cos \omega t} = V_0 \cancel{\cos \omega t}$ time dependence drops out!

Algebraic eq: $Q_0 = -\frac{V_0}{L \omega^2}$

So: $Q(t) = -\frac{V_0}{L \omega^2} \cos \omega t$

$$I(t) = \frac{V_0}{L \omega} \sin \omega t \quad \checkmark$$

Ansatz still works with C but what about $R \dot{Q}$?

Want function that goes back to itself after 1 derivative:

Exponential!

But e^t, e^{-t} blow up or go to 0, expect periodic: $e^{i \omega t}$?

Approach:

- Generalize drive: $V = V_0 e^{i \omega t}$
- Ansatz: $Q = Q_0 e^{i \omega t}$

Since Kirchhoff's law is real and linear, if we have:

$$L \ddot{Q} + R \dot{Q} + Q/C = V_0 e^{i \omega t}$$

Then take the real part of both sides:

$$\text{Re}[L \ddot{Q} + R \dot{Q} + Q/C] = \underbrace{L (\text{Re} \ddot{Q}) + R (\text{Re} \dot{Q}) + \frac{1}{C} (\text{Re} Q)} = V_0 \cos \omega t$$

Thus, $\text{Re} Q$ solves our original problem.

Plugging in $\mathbb{C}$ ansatz:

$$Q = Q_0 e^{i\omega t}$$

$$\dot{Q} = Q_0 (i\omega) e^{i\omega t}$$

$$\ddot{Q} = Q_0 (i\omega)^2 e^{i\omega t}$$

$$\therefore -L Q_0 \omega^2 e^{i\omega t} + R Q_0 i\omega e^{i\omega t} + Q_0 e^{i\omega t} / C = V_0 e^{i\omega t}$$

So:

$$Q_0 = \frac{V_0}{-\omega^2 L + i\omega R + 1/C}$$

$$Q(t) = \frac{V_0 e^{i\omega t}}{-\omega^2 L + i\omega R + 1/C}$$

The current:
$$I(t) = \frac{V_0 e^{i\omega t}}{i\omega L + R + 1/i\omega C}$$

$$I(t) = \frac{V_0 e^{i\omega t}}{Z} \quad \text{Complex Ohm's Law}$$

complex impedance $Z = i\omega L + R + 1/i\omega C$

Actual current:
$$\text{Re } I(t) = \text{Re } \frac{V_0 e^{i\omega t}}{Z}$$

Now write
$$Z = |Z| e^{i\phi} \quad \text{in polar form}$$

$$\text{Re } I(t) = \text{Re } \frac{V_0 e^{i\omega t}}{|Z| e^{i\phi}} = \frac{V_0}{|Z|} \text{Re } e^{i\omega t - i\phi}$$

$$= \frac{V_0}{|Z|} \cos(\omega t - \phi)$$

amplitude

phase shift

Here:

$$|Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

$$\tan \phi = \frac{\omega L - 1/\omega C}{R}$$

