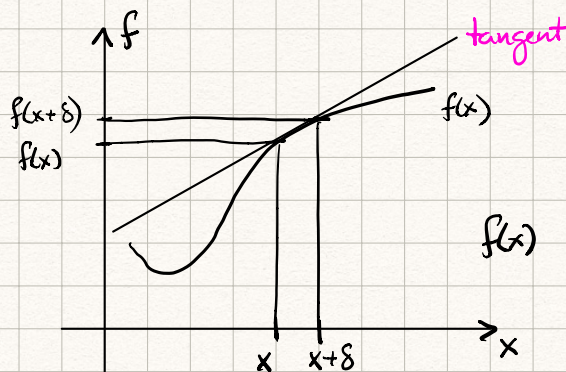


Differential Calculus of 1 Real Variable

cf AVO CI
Shankar CI

Why derivatives? Quantify smooth changes locally.



The derivative of f is well-defined at x if $f(x)$ can be approximated by a linear function near x :

linear function of δx

$$f(x + \delta x) = f(x) + \delta f \approx f(x) + f'(x)\delta x + O(\delta x^2)$$

small deviation from x

derivative @ x

terms of order δ^2 and higher dropped

If $\delta = 10^{-2}$, $\delta^2 = 10^{-4}$ so neglect should be OK.

Ex $f(x) = x^3$

$$\begin{aligned} f(x + \delta x) &= (x + \delta x)^3 = (x + \delta x)(x + \delta x)(x + \delta x) = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3 \\ &= f(x) + 3x^2\delta x + O(\delta x^2) \end{aligned}$$

$$\therefore f'(x) = \frac{\delta f}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} \rightarrow 3x^2$$

Say quantity δx is infinitesimal if we work with it planning to take $\delta x \rightarrow 0$ at the end of a calculation.

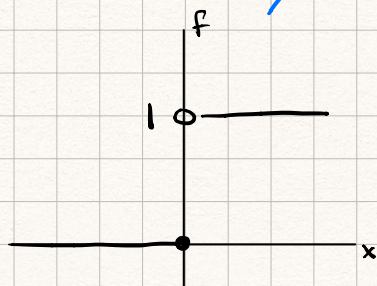
Formally:

$$\underbrace{\frac{df(x)}{dx}}_{\text{derivative}} \equiv \lim_{\delta x \rightarrow 0} \underbrace{\frac{\delta f}{\delta x}}_{\text{difference quotient} \rightarrow \text{slope}} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Say f is differentiable at x if limit exists.

Notation: $f' = \frac{df}{dx} = d_x f = f_x$ equivalent notations for the derivative rep
 $f'(x) = \frac{df(x)}{dx} = d_x f(x) = f_x(x) = \left. \frac{df}{dy} \right|_{y=x}$ notations for value @ x

Differentiability

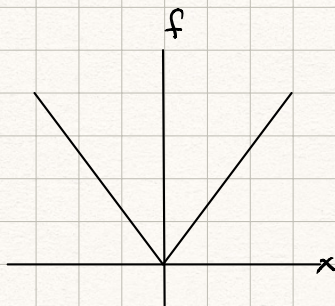


$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

not continuous at $x=0$

so definitely no derivative

$$\frac{f(\delta) - f(0)}{\delta} = \frac{1}{\delta} \rightarrow \infty$$

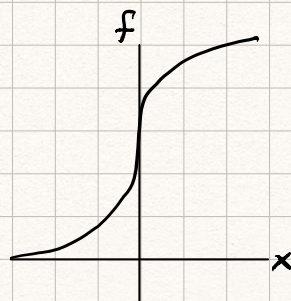


$$f(x) = |x|$$

continuous but
not differentiable @ 0

$$f'(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$

But limit ambiguous
at 0



$$f(x) = 1 + x^{1/3}$$

vertical @ $x=0$
slope = ∞

$$f'(x) = \frac{1}{3} x^{-2/3} \rightarrow \infty \text{ as } x \rightarrow 0$$

Differentiation Rules

Linearity: $\frac{d}{dx}(af(x) + bg(x)) = a \frac{df}{dx}(x) + b \frac{dg}{dx}(x) \quad (a, b \in \mathbb{R})$

Product Rule: $\frac{d(fg)}{dx} = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$

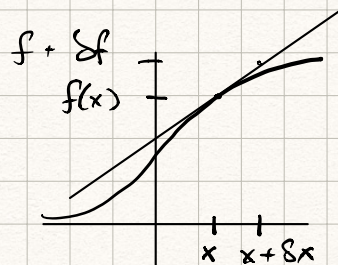
Chain rule: $\frac{df(g(x))}{dx} = \frac{df(y)}{dy} \bigg|_{y=g(x)} \frac{dg(x)}{dx}$

Why? $f(g(x+\delta)) \stackrel{\uparrow}{\approx} f(g(x) + g'(x)\delta) \stackrel{\uparrow}{\approx} f(g(x)) + f'(g(x)) \cdot g'(x)\delta$
 approx g by linear func near x approx f by linear func near $g(x)$

Inverse Rule:

$$\frac{df}{dx} \frac{dx}{df} = 1$$

slope of f as func of x slope of x as func of f



More formally: $\frac{df^{-1}(y)}{dy} = \frac{1}{\frac{df(x)}{dx} \bigg|_{x=f^{-1}(y)}}$

Higher derivatives:

$$\frac{d^2 f(x)}{dx^2} = \frac{d}{dx} \left(\frac{df(x)}{dx} \right)$$

Apply the derivative operator $\frac{d}{dx}$ twice

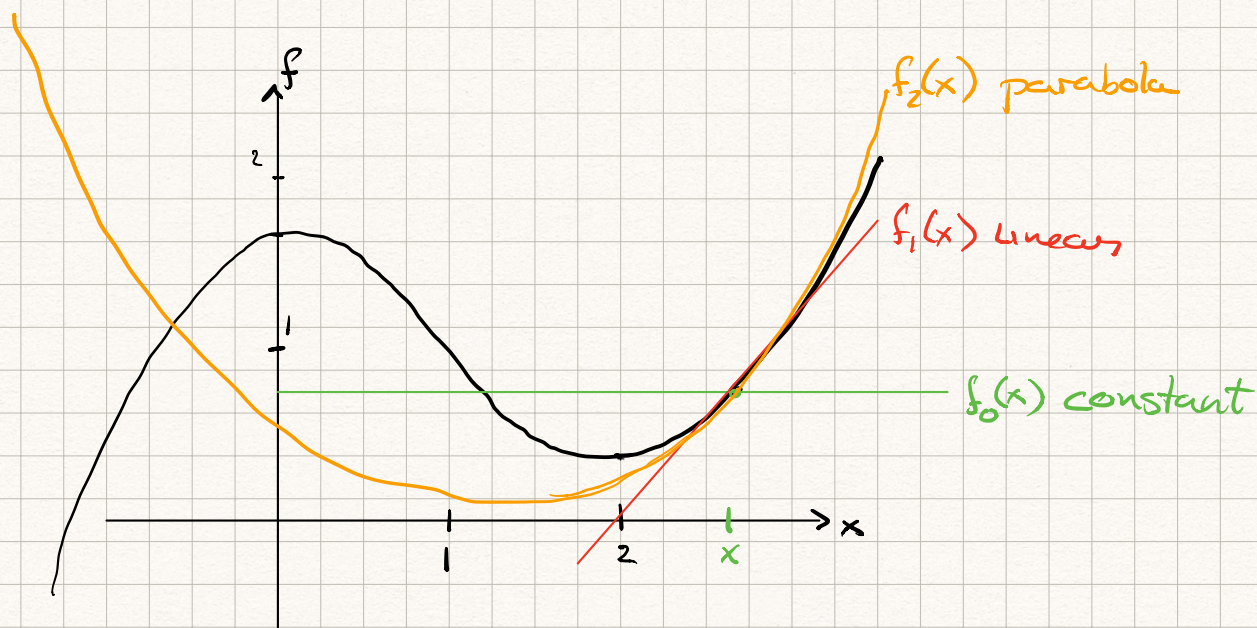
Notation: $f^{(n)}(x) = d_x^n f(x) = \frac{d^n f(x)}{dx^n}$

A function is smooth in an open set (a, b) (or $\mathbb{R}$) if all derivatives exist.

Q? Interpretation of higher derivatives as approximations?

Taylor expansion: match up to n^{th} order in δx by matching derivatives at x up to n^{th}

$$f(x + \delta x) \approx f(x) + f'(x) \delta x + \frac{f''(x)}{2!} \delta x^2 + O(\delta x^3)$$



If f is smooth can write Taylor series w/ ∞ terms

$$f(x) \stackrel{?}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (\delta x)^n$$

Q? Does this series have to equal $f(x)$ on domain of f ?

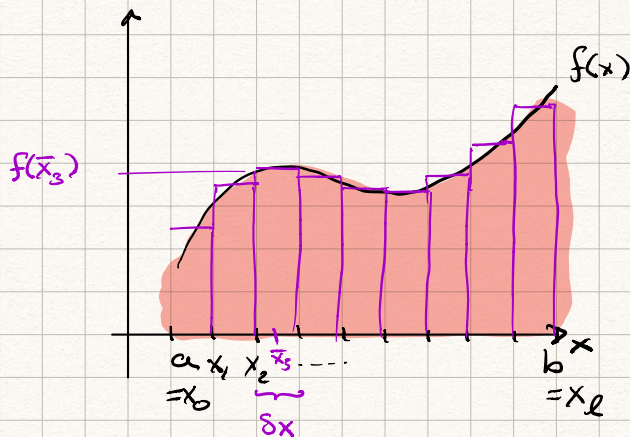
No. 1) Series may not converge.

Ratio test, etc

2) f could be defined with pieces

Integration

Idea: divide domain into lots of little pieces and sum local contribution



To estimate area under curve $f(x)$ from a to b :

Divide (a, b) into l bins of width

$$\Delta x = \frac{b-a}{l} = x_i - x_{i-1}$$

Draw a rectangle of height $f(\bar{x}_i)$

any point in bin i will do
Q? why?

$$\text{Area} \approx \sum_{i=1}^l \Delta x \cdot f(\bar{x}_i) \leftarrow \text{Riemann sum}$$

Becomes exact as $l \rightarrow \infty$ and bin size $\Delta x_i \rightarrow 0$:

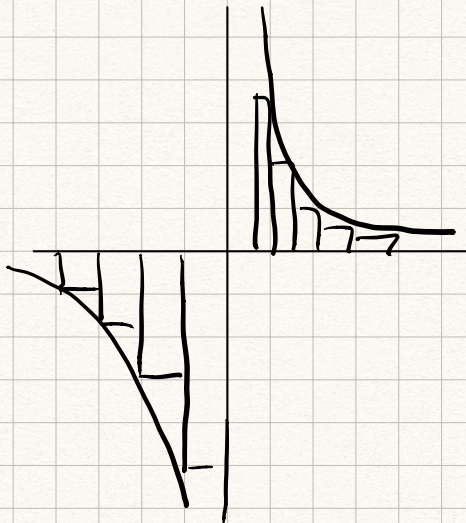
$$\int_a^b dx f(x) \equiv \lim_{\Delta x \rightarrow 0} \sum_{i=1}^l \Delta x f(\bar{x}_i)$$

definite integral over domain (a, b)

Riemann Integral

$\int$ is just a weird s standing for sum

Integral exists so long as limit exists independent of way bins are chosen.



Continuous functions are integrable.

Divergences/singularities in $f(x)$ may or may not make sum ill-defined:

$$\int_0^1 \frac{dx}{x} = \ln x \Big|_0^1 = \ln 1 - \lim_{x \rightarrow 0^+} \ln x = -\infty$$

Not integrable across singularity @ $x=0$.

$$\int_0^1 \frac{dx}{x^{1/2}} = 2x^{1/2} \Big|_0^1 = 2 - 0 = 2$$

Fundamental theorem of calculus

Integrals and derivatives are "inverse" operations (up to constants)

$$\boxed{\int_a^b f'(x) dx = f(x) \Big|_a^b = f(b) - f(a)} \quad \int \frac{d}{dx} \text{ version}$$

$$\underbrace{\sum f'(x) \delta x}_{\uparrow \delta x \rightarrow 0} \approx \sum \delta f_i + O(\delta x^2) = f(b) - f(a) + O(\delta x)$$

The sum of local changes in f is the total change in f .

Alternatively, integral as function of end point is an *anti-derivative*:

$$F(x + \delta x) = \int_a^{x + \delta x} f(y) dy = \int_a^x f(y) dy + \int_x^{x + \delta x} f(y) dy$$

$$\approx F(x) + \underbrace{f(x) \delta x}_{\sim \text{area of last } \delta x \text{ bin}} + O(\delta x^2)$$

$$\therefore \boxed{\frac{dF(x)}{dx} = \frac{d}{dx} \int_a^x f(y) dy = f(x)} \quad \frac{d}{dx} \int \text{ version}$$

Antiderivatives are defined up to *integration constants* C .

$$\frac{d}{dx} (F(x) + C) = \frac{dF(x)}{dx} + \frac{dC}{dx} \overset{0}{=} f(x)$$

Corresponds to choice of lower limit of domain above.

Can evaluate any definite integral if you know an antiderivative using version I:

$$\int_a^b f(x) dx = F(x) \Big|_a^b$$

Tables summarize antiderivatives as "indefinite integrals"

$$\int dx \frac{1}{x} = \ln x \quad \Leftrightarrow \quad \frac{d}{dx} \ln x = \frac{1}{x}$$

↑ indefinite

Ex

$$\int_1^2 \frac{dx}{x} = \ln x \Big|_1^2 = \ln 2 - \ln 1 = \ln 2$$

Integration rules

Just a few important ones. Follow from derivative rules.

Linearity: $\int a f(x) + b g(x) dx = a \int f(x) dx + b \int g(x) dx$

Integration by parts:

$$\frac{d}{dx}(uv) \underset{\substack{\uparrow \\ \text{product rule}}}{=} \frac{du}{dx}v + u \frac{dv}{dx} \Rightarrow \underbrace{\int_a^b dx \frac{d}{dx}(uv)}_{uv \Big|_a^b} = \int_a^b dx \frac{du}{dx} v + \int_a^b dx u \frac{dv}{dx}$$

Rearrange:

$$\boxed{\int_a^b dx u'(x) v(x) = u(x) v(x) \Big|_a^b - \int_a^b dx u(x) v'(x)}$$

Surprisingly useful.

Ex $\int_0^\infty dx x e^{-x} = \underbrace{-e^{-x} x}_{u=-e^{-x}, v=x} \Big|_0^\infty - \int_0^\infty dx -e^{-x} \cdot 1 = \int_0^\infty e^{-x} dx = -e^{-x} \Big|_0^\infty = 1$
 $u' = e^{-x}, v' = 1$

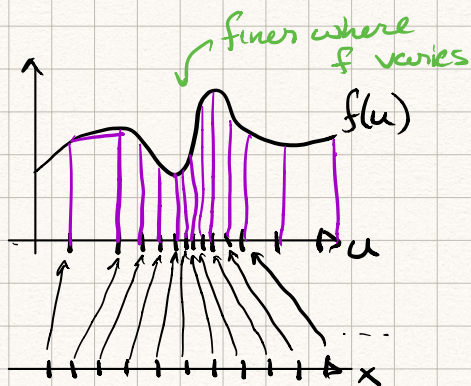
u-substitution:

Let $u(x)$ be monotonically increasing so $a < b \Rightarrow u(a) < u(b)$

$$\int_a^b dx \frac{du(x)}{dx} f(u(x)) = \int_{u(a)}^{u(b)} du f(u) \quad \text{follows from chain rule}$$

Ex $\int_0^\infty dx x e^{-x^2} = \int_0^\infty \frac{du}{2} e^{-u} = \frac{1}{2}$
 $u = x^2$
 $\frac{du}{dx} = 2x$

Intuition is that we can use varying width discretizations:



$$\int_{u_0}^{u_1} f(u) du = \lim_{l \rightarrow \infty} \sum_{j=1}^l (u_j - u_{j-1}) f(u_j)$$

Let $u(x): [a, b] \rightarrow [u_0, u_1]$ such that regular grid in x leads to $u_j = u(x_j)$:

Then

$$\begin{aligned} \int_{u_0}^{u_1} f(u) du &\approx \sum_{j=1}^l (u_j - u_{j-1}) f(u_{j-1}) \\ &= \sum_{j=1}^l (u(x_j) - u(x_{j-1})) f(u(x_{j-1})) \\ &= \sum_{j=1}^l (u(x_{j-1} + \delta x) - u(x_{j-1})) f(u(x_{j-1})) \\ &\approx \sum_{j=1}^l \frac{du}{dx}(x_{j-1}) \delta x f(u(x_{j-1})) \\ &\xrightarrow{l \rightarrow \infty} \int_a^b dx \frac{du}{dx} f(u(x)) \end{aligned}$$

Tips for Integration

Derivatives are mechanical. Antiderivatives are not.

$$\int f(x) dx \stackrel{?}{=}$$

Take practice and some guess work to get antiderivative.

Always check candidate $F(x)$ satisfies $\frac{dF(x)}{dx} = f(x)$.

Strategies:

- Guess and adjust.
- u-substitution
- Integration by parts.
- Table of integrals

→ Antiderivative need not be elementary:

$$\int_0^x e^{-x^2} dx = \operatorname{erf}(x) \quad \text{defines "error function"}$$

Computers can help (Mathematica, Alpha, Maple, etc), but may lose insight into structure of integral.

Definite integrals can be numerically computed by discretization

→ HW.