

Differential Equations

Physical law is local — particles move smoothly in space and time according to what's nearby. Hence, most physics is described by differential equations.

Ex Newton's laws (ODE)



$$\vec{F} = m\vec{a} = m \frac{d\vec{v}}{dt}$$

eg. gravity: $\frac{d\vec{v}}{dt} = -g\hat{z}$

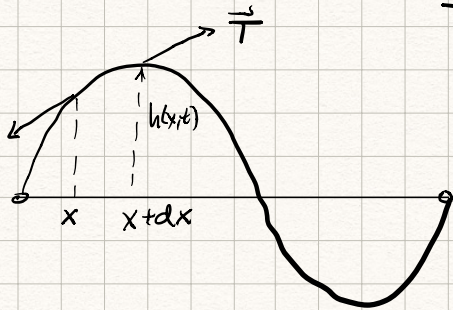
Relates velocity of particle at nearby times:

$$\vec{v}(t+dt) = \vec{v}(t) + dt \cdot \frac{d\vec{v}}{dt} = \vec{v}(t) + dt(-g\hat{z})$$

And position:

$$\vec{x}(t+dt) = \vec{x}(t) + dt \frac{d\vec{x}}{dt} = \vec{x}(t) + dt \vec{v}(t)$$

Ex Waves on a string (PDE)



$$\frac{\partial^2 h}{\partial t^2} - v^2 \frac{\partial^2 h}{\partial x^2} = 0$$

↑
acceleration of
h in small
dt

↖
shape of h in
region dx

def In context of differential equations, function being differentiated $h(x,t)$

is called the dependent variable while coordinates x, t are the independent variables.

If 1 independent variable → ordinary differential equation (ODE)

If >1 " → partial differential equation (PDE)
(relates partial derivatives)

def The **order** of a term is the highest derivative appearing.
 The **degree** of a term is the power of the dependent var.
 The **order/degree** of a DE is the largest order/degree of any term.

Ex $m \frac{dv}{dt} = mg - bv^2$ (falling w/ drag)

Order	$\boxed{1}$	0	0	} $\Rightarrow$ first order, 2nd degree ODE
Degree	1	0	$\boxed{2}$	

$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 \nabla^2 \psi}{2m} - \frac{e^2}{r} \psi$ (Schrödinger Atom)

Order	1	$\boxed{2}$	1	$\Rightarrow$ 2nd order, 1st deg PDE
Degree	1	$\boxed{1}$	1	

$\frac{\partial^2 \phi}{\partial t^2} - v^2 \frac{\partial^2 \phi}{\partial x^2} + \sin \phi = 0$ (Sine-Gordon Egn)

Order	2	2	0	$\Rightarrow$ 2nd order, nonlinear PDE
Degree	1	1	not a simple power!	

def A **linear** DE is constructed from a linear diff. operator:

$\mathcal{L} h(x,t) = g(x,t)$

linear op $\nearrow$ $\mathcal{L}$ $\nwarrow$ dependent (unknown) $\nwarrow$ independent variables $\nwarrow$ "source", known $\nwarrow$ $g(x,t)$

Ex Gravity: $\mathcal{L} = m \frac{d}{dt}$ $g(t) = -mg$ (for v_z)

Waves: $\mathcal{L} = \frac{\partial^2}{\partial t^2} - v^2 \frac{\partial^2}{\partial x^2}$ $g(x,t) = 0$

Q! Is $x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + e^{-x}$ a linear operator?

Prop DE is linear $\Leftrightarrow$ all terms have degree $n=1$ or 0

def An DE is homogeneous if all terms have same degree:

$$\frac{d}{dx} (f(x))^2 + f(x)^2 = 0$$

Else inhomogeneous. Most important case is linear homogeneous:

$$\boxed{\mathcal{L} f(x) = 0}$$

Props - DEs can have multiple general solutions

for n^{th} order ODE, typically n parameters

- In applications, specific solutions fixed by boundary conditions.

- For linear homogeneous, superpositions / linear combinations of solutions are solutions:

$$\mathcal{L} f_1(x) = 0 \quad \mathcal{L} f_2(x) = 0$$

$$\Rightarrow \mathcal{L}(a f_1(x) + b f_2(x)) = a \mathcal{L} f_1(x) + b \mathcal{L} f_2(x) = 0$$

- Useful to identify linearly independent set to form a basis.
(will return to this)

- No general method for solving all DEs.

- Most general: numerical integration

- Many important classes have analytic solutions

- Often associated w/ special functions (Legendre polys)

- Best to recognize type of equation and look up solutions

Rule: Always check your solutions by plugging back in.

However you get a solution, if it plugs in and checks,
you win!

This class: ODEs only — no time for even that
let alone PDEs.

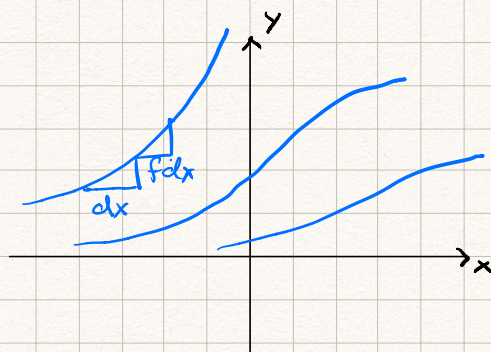
First order ODEs

- Simplest kind of ODE. Always solvable (see books)
- General form:

$$\frac{dy}{dx} = f(x, y)$$

↑ dependent
↑ independent

slope of $y(x)$ if
 $y(x)$ goes through
point (x, y)



- Useful to rationalize (may be multiple choices of P, Q)

$$\frac{dy}{dx} = - \frac{P(x, y)}{Q(x, y)}$$

Separable ODEs

The ODE is separable if

$$\frac{dy}{dx} = \frac{P(x)}{Q(y)}$$

Then: $P(x)dx = Q(y)dy$

Integrate from (x_0, y_0) to (x, y) :

$$\int_{x_0}^x P(x') dx' = \int_{y_0}^y Q(y') dy'$$

Ex decay of radioactive isotope with rate Γ

$$\frac{dN}{dt} = -\Gamma N$$

Separate: $\frac{dN}{N} = -\Gamma dt$

$$\int_{N_0}^N \frac{dN}{N} = -\Gamma \int_0^t dt \Rightarrow \ln N = -\Gamma t + C$$

General solution:

$$N(t) = C'e^{-\Gamma t}$$

← fixed by boundary conditions

If $N(t=0) = N_0$, then

$$N(t) = N_0 e^{-\Gamma t}$$

← specific solution

Ex The Parachutist



Newton's Law:

downward velocity

gravity drag

$$m\dot{v} = mg - bv^2$$

①

Q? linear? homogeneous?

Terminal velocity? $0 = mg - bv^2 \Rightarrow v_0 = \sqrt{\frac{mg}{b}}$

Simplify ①: $\frac{m}{b} \ddot{v} = v_0^2 - v^2$

Separate: $\frac{m}{b} \frac{dv}{dt} = v_0^2 - v^2 \Rightarrow \frac{dv}{v_0^2 - v^2} = \frac{b}{m} dt$

Integrate: $\int_0^{v(t)} \frac{dv}{v_0^2 - v^2} = \int_0^t \frac{b}{m} dt = \frac{b}{m} t$

$$= \int_0^{v(t)} dv \frac{1}{(v_0 - v)(v_0 + v)} = \int_0^{v(t)} \frac{dv}{2v_0} \left[\frac{1}{v_0 - v} + \frac{1}{v_0 + v} \right]$$

$$= \frac{1}{2v_0} \left[-\ln(v_0 - v) + \ln(v_0 + v) \right]_0^{v(t)}$$

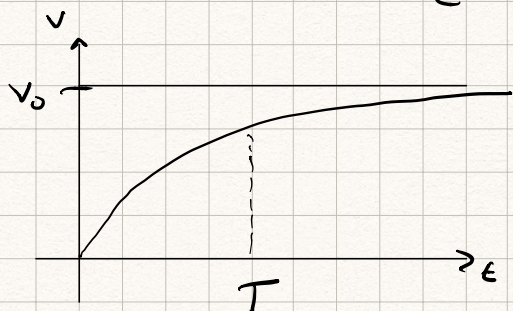
$$\therefore \frac{1}{2v_0} \ln \frac{v_0 + v}{v_0 - v} = \frac{b}{m} t$$

$$\frac{v_0 + v}{v_0 - v} = e^{2v_0 \frac{b}{m} t}$$

$$T^{-1} = v_0 \frac{b}{m}$$

↑ time to approach v_0

$$v = v_0 \frac{e^{2t/T} - 1}{e^{2t/T} + 1} = v_0 \tanh t/T$$



Check: $\frac{d}{dx} \tanh x = 1 - \tanh^2 x$

$$\ddot{v} = \frac{v_0}{T} (1 - \tanh^2 t/T)$$

$$= \frac{v_0}{T} - \frac{v^2}{Tv_0} = g - \frac{b}{m} v^2 \quad \checkmark$$