

Hermitian Eigensystems

Most important eigenproblems in physics involve

Hermitian operators:

def The Hermitian conjugate / adjoint of $\hat{A}$ is the operator $\hat{A}^+$ such that $\langle v, \hat{A}w \rangle = \langle \hat{A}^+v, w \rangle \quad \forall v, w \in \mathcal{H}$

Prop Matrix on ON basis: $A^+ = A^{T*}$

def If $A^+ = A$, A is Hermitian / self-adjoint.

Spectral Thm

If A is Hermitian operator on Hilbert space $\mathcal{H}$:

- 1) Eigenvalues λ_i are real.
- 2) Eigenvectors $|v_i\rangle$ corresponding to distinct λ_i are orthogonal
- 3) Possible to choose eigenvectors to be a complete orthonormal basis for $\mathcal{H}$, called the eigenbasis

Comments

- Set of eigenvalues of A is called spectrum of A
- For $n \times n$ A , $p_A(\lambda) = \det(A - \lambda I)$ is n^{th} degree poly,
→ n roots counting multiplicity
- Completeness $\Rightarrow$ degenerate root ω / multiplicity k
has k -dimensional degenerate eigenspace
- Spectral thm also holds in infinite dimensional function spaces
 - Good way to find ON bases for such spaces

Ex

$$A = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

Q? Hermitian?

Secular Eqn:
$$P_A(\lambda) = \begin{vmatrix} -\lambda & & 1 \\ & 1-\lambda & \\ 1 & & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) - (1-\lambda)$$
$$= (\lambda^2 - 1)(1-\lambda) = (\lambda+1)(\lambda-1)(1-\lambda)$$

Spectrum:
$$\begin{cases} \lambda_1 = -1 & \text{multiplicity } 1 \\ \lambda_2 = +1 & \text{" } 2 \end{cases}$$

Eigenvectors:

$$\lambda_1 = -1: \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ b \\ a \end{pmatrix} \stackrel{!}{=} - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow v_1 = c \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_2 = +1: \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ b \\ a \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\left. \begin{matrix} c = a \\ b = b \\ a = c \end{matrix} \right\} \begin{matrix} 2 \text{ linear dependencies} \\ \text{b/c mult} = 2 \end{matrix}$$

$$v_2 = \begin{pmatrix} c_1 \\ c_2 \\ c_1 \end{pmatrix} \quad 2\text{-D eigenspace}$$

Can find orthogonal basis for the degenerate eigenspace by inspection or by Gram-Schmidt (see HW).

Normalizing: $\lambda_1 = -1$

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\lambda_2 = +1$

$$v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Complete orthonormal basis for $\mathcal{H} \leftarrow$ eigenbasis of A

proof of ① (eigenvalues real)

Consider $A|v\rangle = \lambda|v\rangle$ ($|v\rangle \neq 0$)

$$\langle v|A|v\rangle = \lambda \langle v|v\rangle \quad (\text{inner prod w/ } \langle v|)$$

Conjugate LHS:

$$\begin{aligned} \langle v|A|v\rangle^* &\stackrel{\text{notation}}{=} \langle v, Av\rangle^* \stackrel{\text{conjugate symmetry}}{=} \langle Av, v\rangle \\ &\stackrel{\text{adjoint}}{=} \langle v, A^\dagger v\rangle \stackrel{\hat{A} \text{ is Hermitian}}{=} \langle v, Av\rangle \\ &= \langle v|A|v\rangle \end{aligned}$$

$\therefore$ LHS is real.

$$\langle v|v\rangle > 0 \quad \text{so} \quad \lambda = \frac{\langle v|Av\rangle}{\langle v|v\rangle} \in \mathbb{R}. \quad \square$$

proof of ② (eigenvectors of distinct evals $\perp$)

$$A|v_1\rangle = \lambda_1|v_1\rangle \quad A|v_2\rangle = \lambda_2|v_2\rangle$$

$$\textcircled{1} \quad \langle v_2|A|v_1\rangle = \lambda_1 \langle v_2|v_1\rangle \quad (\langle v_1|A|v_2\rangle)^* = (\lambda_2 \langle v_1|v_2\rangle)^*$$

$$\textcircled{2} \quad \langle v_2|A|v_1\rangle = \lambda_2 \langle v_2|v_1\rangle$$

$$\text{Subtract } \textcircled{1} \text{ } \& \textcircled{2}: \quad 0 = (\lambda_1 - \lambda_2) \langle v_2|v_1\rangle$$

$$\text{So if } \lambda_1 \neq \lambda_2, \quad \langle v_2|v_1\rangle = 0 \quad \square$$

Convent If spectrum is non-degenerate, n evals $\Rightarrow n \perp$ evects
 $\rightarrow$ $\textcircled{3}$ eigenbasis is complete.

Case with degeneracy is slightly harder to prove.

See Wikipedia for sketch.

Diagonalization

Say $\hat{A}$ has eigenbasis $|v_i\rangle$:

$$\hat{A}|v_i\rangle = \lambda_i |v_i\rangle$$

What is matrix of $\hat{A}$ in eigenbasis? **Diagonal!**

$$\hat{A}|v_j\rangle = \sum_i |v_i\rangle A_{ij} = \sum_i |v_i\rangle \lambda_j \delta_{ij}$$

$$\text{So } A_{ij} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

What if $\hat{A}$ is given by matrix A_y in standard (or) basis?

① Solve $A \bar{v}_i = \lambda_i \bar{v}_i$ in standard basis

→ Get column vectors $\bar{v}_i$ (choose ON)

② Change basis:

$$T^\dagger = \left(|v_1\rangle |v_2\rangle |v_3\rangle \dots |v_n\rangle \right)$$

↑
basis transformation
Eigenbasis → Standard

$$D = T A T^\dagger$$

↑
diagonal ($\hat{A}$ in eigenbasis)

original basis

Note: Since T transforms ON to ON, it is **unitary**

$$T^\dagger = T^{-1}$$

Check

$$D \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad T A T^\dagger \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = T A \bar{v}_1 = \lambda_1 T \bar{v}_1 = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

So solving eigensystems also called diagonalization.

→ Eigensolvers (computer) return spectrum and T (or $T^\dagger$).

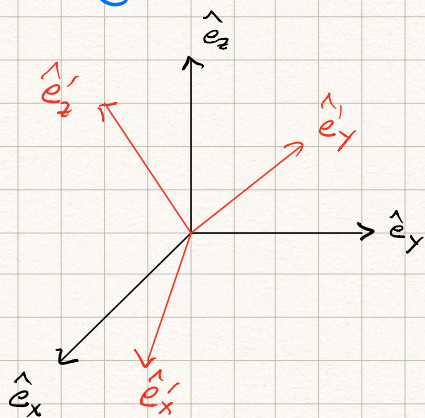
Ex

$$A = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

$$T^+ = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{aligned} T A T^+ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = D \end{aligned}$$

Orthogonal Transformations



An **orthogonal transformation** is a linear transformation which **preserves** the length of any vector $\vec{v}$ and the angle between any pair $\vec{v}, \vec{w}$.

$$\vec{v}' = S\vec{v}$$

Consider unit vectors $\hat{e}_i$:

$$\vec{e}'_i = S\vec{e}_i \quad i^{\text{th}} \text{ column}$$

$$S = \left(\begin{array}{c|c|c} s_1 & s_2 & \dots & s_n \end{array} \right)$$

To preserve length & angles:

$$s_i^T s_j = (S\vec{e}_i)^T (S\vec{e}_j) = \vec{e}_i^T S^T S \vec{e}_j \stackrel{!}{=} \vec{e}_i^T \vec{e}_j = \delta_{ij}$$

- the column vectors of S form an orthonormal basis.

def The matrix S is an **orthogonal matrix** if $S^T = S^{-1}$.

$$(\text{i.e. } S^T S = S S^T = \mathbb{1})$$

Orthogonal matrices represent orthogonal transformations.

Ex Show $S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is orthogonal.

$$S^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad S^T S = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \mathbb{1}$$

- $\vec{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ has length $|\vec{v}| = \sqrt{3^2 + 4^2} = 5$

$$\text{Check: } S\vec{v} = \frac{1}{\sqrt{2}} \begin{pmatrix} 7 \\ 1 \end{pmatrix} \quad |S\vec{v}| = \frac{1}{\sqrt{2}} \sqrt{49 + 1} = 5 \quad \checkmark$$

Props of Orthogonal Transformations / Matrices

(closed under product) S, S' orthogonal $\Rightarrow SS'$ is orthogonal

$$(SS')^T(SS') = S'^T S^T SS' = S'^T S^{-1} SS' = \mathbb{1}$$

In terms of transformations this is obvious:

If S and S' preserve lengths, then first doing S' and then S will preserve lengths.

(closed under inverse) S orthogonal $\Rightarrow S^{-1} = S^T$ is orthogonal

(identity is orthogonal) $\mathbb{1}$ preserves lengths. Also, $\mathbb{1}^T = \mathbb{1} = \mathbb{1}^{-1}$

The set of $n \times n$ orthogonal matrices is called the orthogonal group $O(n)$.

(determinant) $1 = \det S^T S = (\det S^T)(\det S) = (\det S)^2$

$$\therefore \det S = \pm 1$$

The orthogonal transformations with

$$\det S = +1$$

are

"proper"

↖ rotations

$$= -1$$

are

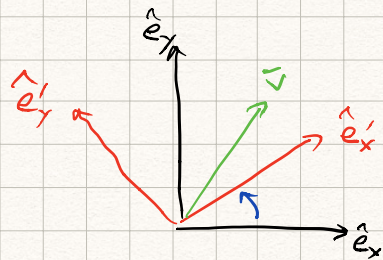
"improper"

↖ reflections

↪ change right handed to left handed basis

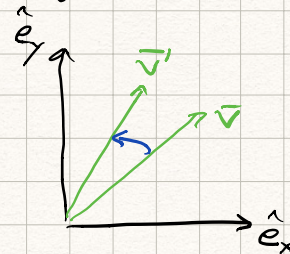
Active vs Passive Transformations

Passive transformation describe how components of a fixed vector transform under change of basis / coordinate system.



Passive

vs



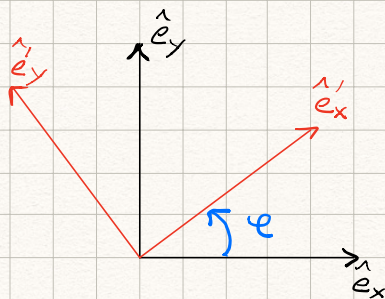
Active

Active transformations change vector in fixed basis.

Matrix representation: $S_{\text{ACTIVE}} = S_{\text{PASSIVE}}^{-1} = S_{\text{PASSIVE}}^T$

Orthogonal Transformations in 2D: $O(2)$

Rotations



- Rotate basis by θ from $\hat{e}$ to $\hat{e}'$

$$S = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

Q?

$$\det S = \cos^2 \theta + \sin^2 \theta = 1$$

Proper transformation

In general, rotations are smoothly connected to $\mathbb{I}$ (rotation angle $\rightarrow 0$), so $\det S = +1$ by continuity.

Reflections



$$S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Q? Orthogonal? $\checkmark$

Q? $\det S = -1$ $\checkmark$

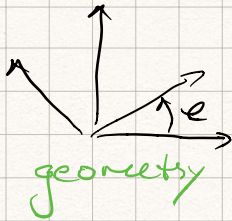
Improper. Changes right-handed to left-handed basis.

Q? Is inversion $v \rightarrow -v$ proper or improper in 2D?

Orthogonal Transformations in 3D: $O(3)$

Rotations ($SO(3)$)

In 2d, rotations parametrized by single angle θ .



$$S^T S = \mathbb{1} \quad \leftarrow \quad \begin{array}{rcl} 4 & \text{unknowns in } S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ - 3 & \text{equations} & (12 \text{ and } 21 \text{ equations} \\ & 1 & \text{parameter} & \text{redundant by symmetry}) \\ \hline & & 1 & \text{parameter} \end{array}$$

counting unconstrained parameters in S

In 3d, 3 independent parameters:

$$S^T S = \mathbb{1} \quad \begin{array}{rcl} 9 & \text{unknowns in } S \\ - 6 & \text{equations} \\ \hline 3 & \text{parameters} \end{array}$$

Q? In n -dimensions, how many params?

$$S^T S = \mathbb{1} \quad \begin{array}{rcl} n^2 & \text{unknowns} \\ - n(n+1)/2 & \text{equations} \\ \hline n(n-1)/2 & \text{parameters} \end{array}$$