

Eigenvalue Problem

Many problems in physics can be reduced to:

Linear operator $\hat{A}$ $\vec{v}$ = $\lambda \vec{v}$

vector in some Hilbert space $\mathcal{H}$

known $\hat{A}$ unknown λ scalar in $\mathbb{R}$ or $\mathbb{C}$

Goal: Given $\hat{A}$, determine

eigenvalues λ

eigenvectors $\vec{v}$ (aka eigenfunctions if $\mathcal{H}$ is a function space)

"Eigen" means "its own" in German:

$\hat{A}$ eats eigenvector $\vec{v}$, produces $\lambda \vec{v}$ in same direction.

Ex - Resonant waves on a string or drumhead:

$\hat{A}h = \lambda h$

force displacement

$F(x) = \underbrace{(-k \nabla^2)}_{\hat{A}} \underbrace{h(x)}_{\vec{v}}$

"normal modes"

- Angular momentum of rigid body

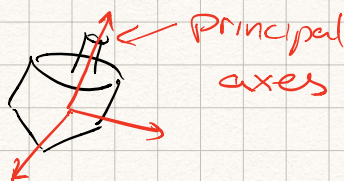
$\vec{L} = \vec{I} \vec{\omega}$

moment of inertia matrix $I = \begin{pmatrix} & \\ & \end{pmatrix}$

Principal axes are those for which $\vec{L} = \lambda \vec{\omega}$

$\vec{I} \vec{\omega} = \lambda \vec{\omega}$

Top:



- Time independent Schrodinger eqn:

$\hat{H} |\psi\rangle = E |\psi\rangle$

Hamiltonian Energy Eigenstate

Matrix Eigenvalue Problems

Can always pick a basis of vector space $\mathcal{H}$

$$\hat{A} \vec{v} = \lambda \vec{v} \rightarrow A \vec{v} = \lambda \vec{v}$$

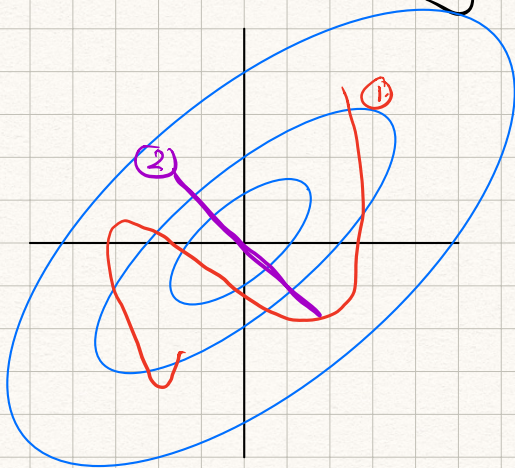
$\uparrow \quad \uparrow$
matrix column vector

If $\mathcal{H}$ has finite dimension $n \Rightarrow$ matrix eigenproblem

Lots of numerical techniques (see HW)

We will develop techniques by example.

Ex Particle Sliding on Ellipsoidal Basin in 2D



Particle @ (x, y)

$$V(x, y) = ax^2 + bxy + cy^2$$

Are there any special initial positions such that particle simply oscillates on a straight line through origin?

force: $\vec{F} = -\nabla V = (-2ax - by)\hat{x} + (-bx - 2cy)\hat{y}$

Want force toward origin: $\vec{F} \propto \vec{r} = x\hat{x} + y\hat{y}$

Typically not. Search for special positions where this holds:

$$\vec{F} = \lambda \vec{r}$$

$\nwarrow$ unknown proportionality constant

$$\begin{pmatrix} -2a & -b \\ -b & -2c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

in standard basis
(same)

$$A \vec{r} = \lambda \vec{r}$$

To solve: ① Find λ which permit solutions

② Find corresponding $\vec{r}$

① Observe: can rearrange $A\bar{r} = \lambda\bar{r}$
 $(A - \lambda I)\bar{r} = 0$

If $(A - \lambda I)$ is nonsingular (invertible), then

$$(A - \lambda I)^{-1}(A - \lambda I)\bar{r} = (A - \lambda I)^{-1}0 = 0$$

$\therefore \bar{r} = 0$ is only solution

Need to find λ st $A - \lambda I$ is singular for non-zero solutions;

$$\det(A - \lambda I) = 0$$

"characteristic equation"
"secular equation"

def $P_A(\lambda) = \det(A - \lambda I)$ is the characteristic polynomial of A .
(aka secular poly)

Ex for $n=2$, $\det(A - \lambda I) = \begin{vmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{vmatrix}$

$$\begin{aligned} P_A(\lambda) &= (A_{11} - \lambda)(A_{22} - \lambda) - A_{12}A_{21} \\ &= \lambda^2 - (A_{11} + A_{22})\lambda + (A_{11}A_{22} - A_{12}A_{21}) \end{aligned}$$

quadratic polynomial in λ w/ coeffs functions of matrix elements A_{ij}

The roots of $P_A(\lambda)$ are the eigenvalues of A .

Ex Take $a=1$ $b=-\sqrt{5}$ $c=3$

$$A = \begin{pmatrix} -2 & \sqrt{5} \\ \sqrt{5} & -6 \end{pmatrix}$$

$$\begin{aligned} P_A(\lambda) &= \begin{vmatrix} -2 - \lambda & \sqrt{5} \\ \sqrt{5} & -6 - \lambda \end{vmatrix} = \lambda^2 + 8\lambda + 7 \\ &= (\lambda + 1)(\lambda + 7) \end{aligned}$$

So roots/eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -7$

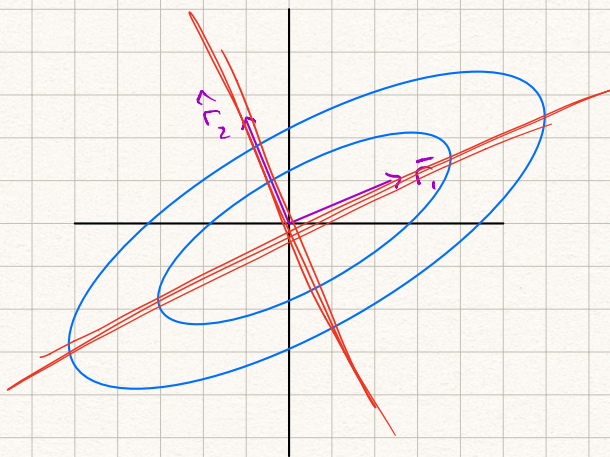
② Find corresponding eigenvectors:

for $\lambda_1 = -1$:

$$(A - (-1)I)\vec{r} = \begin{pmatrix} -1 & \sqrt{5} \\ \sqrt{5} & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{array}{l} -x + \sqrt{5}y = 0 \\ \sqrt{5}x - 5y = 0 \end{array} \right\} \Rightarrow \text{Both solved by } x = \sqrt{5}y$$

So: $\lambda_1 = -1$, $\vec{r}_1 = C \begin{pmatrix} \sqrt{5} \\ 1 \end{pmatrix}$ for any constant C



for $\lambda_2 = -7$:

$$\begin{pmatrix} 5 & \sqrt{5} \\ \sqrt{5} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{array}{l} 5x + \sqrt{5}y = 0 \\ \sqrt{5}x + 1y = 0 \end{array} \right\} \Rightarrow y = -\sqrt{5}x$$

$$\vec{r}_2 = C' \begin{pmatrix} 1 \\ -\sqrt{5} \end{pmatrix}$$

Check: $A\vec{r}_1 = \begin{pmatrix} -2 & \sqrt{5} \\ \sqrt{5} & -6 \end{pmatrix} C \begin{pmatrix} \sqrt{5} \\ 1 \end{pmatrix} = C \begin{pmatrix} -\sqrt{5} \\ -1 \end{pmatrix} = (-1)\vec{r}_1 = \lambda_1 \vec{r}_1 \quad \checkmark$

Normalization: The eigenvalue equation doesn't care about the magnitude of the eigenvectors:

$$A\vec{r}_1 = \lambda_1 \vec{r}_1 \quad \leftrightarrow \quad A(2\vec{r}_1) = \lambda_1 (2\vec{r}_1)$$

That's why C, C' were free above.
Conventional to normalize:

$$1 = \vec{r}_1^2 = C^2(5+1) \Rightarrow C = \frac{1}{\sqrt{6}} \quad \vec{r}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{5} \\ 1 \end{pmatrix}$$

...

$$\vec{r}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -\sqrt{5} \end{pmatrix}$$

Observations

- # of eigenvalues = dimension n of $\mathcal{H}$
 P_A has degree $n \hookrightarrow n$ roots by fundamental theorem of algebra
- Eigenvalues were real (in example) even though quadratic eqn can have $\mathbb{C}$ roots
- Eigenvectors are orthogonal (in example)
- Directions define **principal axes** of potential

Can rotate coordinates by projecting onto principal axes:

$$\bar{r} = x\hat{x} + y\hat{y} = x'\hat{r}_1 + y'\hat{r}_2$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \hat{r}_1 \cdot \bar{r} \\ \hat{r}_2 \cdot \bar{r} \end{pmatrix} = \begin{pmatrix} \sqrt{5}/6 & \sqrt{1}/6 \\ 1/\sqrt{6} & -\sqrt{5}/6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix} \quad \swarrow \text{rotation matrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = R^{-1} \begin{pmatrix} x' \\ y' \end{pmatrix} = R^T \begin{pmatrix} x' \\ y' \end{pmatrix}$$

In these coords:

$$V = x^2 - \sqrt{5}xy + 3y^2$$

$$= (\sqrt{5}/6 x' + \sqrt{1}/6 y')^2 - \sqrt{5}(\sqrt{5}/6 x' + \sqrt{1}/6 y')(\sqrt{1}/6 x' - \sqrt{5}/6 y') + 3(\sqrt{1}/6 x' - \sqrt{5}/6 y')^2$$

$$= \frac{1}{6} \left[5x'^2 + 2\sqrt{5}x'y' + y'^2 - \sqrt{5}(\sqrt{5}x'^2 - 5x'y' + x'y' - \sqrt{5}y'^2) + 3(x'^2 - 2\sqrt{5}x'y' + 5y'^2) \right]$$

$$= \frac{1}{6} [3x'^2 + 21y'^2] = \frac{1}{2}x'^2 + \frac{7}{2}y'^2$$

simple decoupled x', y' springs