

Trace

- Another basis invariant characteristic of a matrix / map

$$\boxed{\text{trace}(A) = \text{tr } A = \sum_{i=1}^n A_{ii}} \quad (\text{sum of diagonal})$$

Props

(linearity) $\text{tr } kA + k'B = k \text{tr } A + k' \text{tr } B$ (follows from defn)

(cyclic property)

$$(*) \text{tr } AB = \text{tr } BA$$

Why? $\text{tr } AB = \sum_i \sum_j A_{ij} B_{ji} = \sum_i \sum_j B_{ji} A_{ij} = \sum_j (BA)_{jj} = \text{tr } BA$

"Cyclic" because (*) implies that

$$\text{tr } ABC = \text{tr } BCA = \text{tr } CAB$$

The trace of a product of matrices is invariant under cyclic permutations of the matrices.

Ex $A = \begin{pmatrix} 1 & 2 \\ & 3 \end{pmatrix} \quad \begin{matrix} \text{tr } A \stackrel{?}{=} 4 \\ \det A \stackrel{?}{=} 3 \end{matrix}$

Transpose

- Transpose A^T of A is obtained by changing rows into columns

Q? $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

reflect across diagonal

Index notation: $(A^T)_{ij} = A_{ji}$ ← swap indices

- A is symmetric if $A^T = A$ and antisymmetric if $A^T = -A$

- Transpose of products $\boxed{(AB)^T = B^T A^T}$

- Rectangular transpose: $m \times n \xrightarrow{T} n \times m$

Ex Column vector $v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $v^T = (1 \ 0 \ 1)$ row vector

$$v^T v = (1 \ 0 \ 1) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2 \quad \text{dot product in matrix notation}$$

Careful: $v v^T = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} (1 \ 0 \ 1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ outer product is $n \times n$

Unit vectors in matrix notation

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \dots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \quad \text{"standard basis"}$$

Can use to extract rows, columns, matrix elements of matrix A:

Column 2: $A e_2 = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b \\ e \\ h \end{pmatrix}$

Row 3: $e_3^T A = (0 \ 0 \ 1) \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = (g \ h \ i)$

element 3,2: $e_3^T A e_2 = (g \ h \ i) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = h = A_{3,2}$

Adjoint / Hermitian conjugate

For matrices with complex $\mathbb{C}$ matrix elements.

Complex conjugate matrix: A^* $(A^*)_{ij} = A_{ij}^*$

Adjoint / Hermitian conjugate: $A^\dagger \equiv A^{*T}$
"dagger"

If $A^\dagger = A$, say A is Hermitian or self-adjoint.

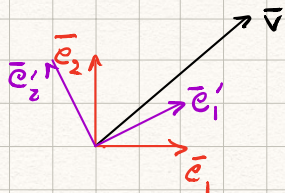
- In quantum: observables represented by Hermitian matrices
- Real symmetric matrices are Hermitian.

Q? $(AB)^\dagger \stackrel{?}{=} B^\dagger A^\dagger$

Changing Basis (from $\bar{e}_i$ to $\bar{e}'_i$)

Recall a vector $\bar{v} \in V$ is some particular "physical" object.

Choose basis of "physical" vectors:



$$\bar{v} = \hat{e}_1 v^1 + \hat{e}_2 v^2 = \sum_i \hat{e}_i v^i = \hat{e}_i v^i$$

Then we can identify component column vectors:

physical vector $\bar{v}$ $\leftrightarrow$ $v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ in $\bar{e}_i$ basis
 has $\uparrow$ column of components has no -

What if we want to work in a different basis $\bar{e}'_i$?

$$\bar{v} = \hat{e}'_1 v'^1 + \hat{e}'_2 v'^2 = \hat{e}'_i v'^i \leftrightarrow v' = \begin{pmatrix} v'^1 \\ v'^2 \end{pmatrix} \text{ in } \bar{e}'_i \text{ basis}$$

We expand the old $\hat{e}_i$ in the new basis $\hat{e}'_i$:

$$\begin{aligned} \hat{e}_1 &= \hat{e}'_1 S_{11} + \hat{e}'_2 S_{21} \\ \hat{e}_2 &= \hat{e}'_1 S_{12} + \hat{e}'_2 S_{22} \end{aligned} \quad \leftrightarrow \quad \bar{e}_j = \bar{e}'_i S_{ij}$$

vector $\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$
 S_{ij} = component of $\hat{e}_j$ in $\hat{e}'_i$ basis
 TR/S-numbers

So:

$$\bar{v} = \hat{e}_i v^i = (\bar{e}'_j S_{ji}) v^i = \bar{e}'_j \underbrace{(S_{ji} v^i)}_{v'^j}$$

Components of $\bar{v}$ transform: $\boxed{v' = S v}$

The transformation matrix S_{ij} describes how components of a vector $\bar{v}$ change when changing basis.

To find S :

$\rightarrow \bar{e}_i \rightarrow e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{\text{th}} \text{ row in old basis}$

$e'_i = S e_i = \begin{pmatrix} s_{1i} \\ \vdots \\ s_{ni} \end{pmatrix}$ i^{th} column of T is expansion of $\bar{e}_i$ in new basis.

$\rightarrow$ So to construct S matrix, just expand old basis vectors new basis.

Inverse change of basis (from $\bar{e}'_i$ to $\bar{e}_i$)

Say components transform

$$v' = S v$$

To transform back, need a matrix that multiplies v' to give v :

$$S^{-1} v' = S^{-1} S v = v$$

So inverse transformation given by matrix inverse of S .

Orthonormal bases

If both bases are orthonormal then S is orthogonal matrix.

Change of Matrix Representation for Linear Transforms

"physical"

Linear map

$$\hat{A}: V \rightarrow V$$
$$\vec{v} \mapsto \vec{w} \equiv \hat{A}\vec{v}$$

"physical" vectors

in basis $\hat{e}_i$

Square matrix

$$A = (A_{ij})$$

$$w = Av$$

in basis $\hat{e}'_i$

Square matrix

$$A' = (A'_{ij})$$

$$w' = A'v'$$

Know how components transform:

$$v = S^{-1}v' \quad w = S^{-1}w'$$

Plug in and multiply by S :

$$w' = SAS^{-1}v'$$

Holds for arbitrary v' so:

$$\boxed{A' = SAS^{-1}}$$

Change of Matrix Repn

similarity transform of A
by S

Read this way:

① S^{-1} : transform to old basis

② A : apply old matrix

③ S : transform back to new basis.

Prop $\det A$ and $\text{tr } A$ are independent of basis choice

Why?