

The Determinant

- basis invariant characteristic of a matrix
- detects if a matrix is singular (ie. not invertible)

$$\det(A) = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{vmatrix} \equiv \sum_{i_1, \dots, i_n} \epsilon_{i_1 \dots i_n} A_{i_1 1} A_{i_2 2} \dots A_{i_n n}$$

Straight Lines $\rightarrow$ $\epsilon_{i_1 \dots i_n}$ is the Levi-Civita symbol (n-dimensional)
 $i_1, \dots, i_n$ are n indices, each goes 1 to n
 $A_{i_1 1}$ is the i_1^{th} row, 1 col
 $A_{i_2 2}$ is the i_2^{th} row, 2 col
 $A_{i_n n}$ is the i_n^{th} row, n col

- The Levi-Civita symbol is completely antisymmetric form in n indices:

$$\epsilon_{i_1 \dots i_n} = \begin{cases} +1 & \text{if } i_1, i_2, \dots, i_n \text{ is an even permutation of } 1, 2, 3, \dots, n \\ -1 & \text{if " " " odd " "} \\ 0 & \text{if any indices are equal} \end{cases}$$

- So, non-zero terms in $\det(A)$:

- for each row, pick an element from a column, but do not use same column twice.

- Sign is ± 1 depending on permutation of columns.

Ex $|A_{ii}| = A_{ii}$

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11} A_{22} - A_{12} A_{21}$$

1 2
2 1

even
odd

Q? How many terms in sum?

$$\det A = \sum_{i_1, \dots, i_n} \epsilon_{i_1, \dots, i_n} A_{i_1, 1} \dots A_{i_n, n} = \sum_{\substack{i_1, \dots, i_n = \\ \text{permutation} \\ \text{of } 1, 2, \dots, n}} \epsilon_{i_1, \dots, i_n} A_{i_1, 1} \dots A_{i_n, n}$$

$$n \circ n \circ n \dots n = n^n$$

$$n(n-1)(n-2) \dots 1 = n!$$

BAD! lots of terms

Still BAD!

- Naive evaluation of $\det A$ is SLOW for big n .
Will return to efficient approaches later.

Expansion in minors (Laplace Rule)

- helps to organize terms in $\det A$ recursively. (still lots)
- useful if matrix is **sparse** - has lots of zeroes.

A **minor** M_{ij} of A is the determinant of $(n-1) \times (n-1)$ submatrix of A obtained by striking out i^{th} row & j^{th} column.

$$M_{ij} = \begin{vmatrix} A_{11} & \dots & A_{1j} & \dots & A_{1n} \\ \vdots & & \vdots & & \vdots \\ A_{i1} & \dots & A_{ij} & \dots & A_{in} \\ \vdots & & \vdots & & \vdots \\ A_{n1} & \dots & A_{nj} & \dots & A_{nn} \end{vmatrix}$$

i^{th} row

j^{th} column

Prop $\det A = \sum_{j=1}^n A_{ij} (-1)^{i+j} M_{ij}$ can expand any row i (or column)

$\uparrow$ $n \times n$ det $\uparrow$ $(n-1) \times (n-1)$ det

can be expanded in $(n-2) \times (n-2)$ det's, etc

Ex $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 6 & 7 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix} - 2 \begin{vmatrix} 0 & 4 \\ 5 & 7 \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 3 \\ 5 & 6 \end{vmatrix}$

$= -3 - 2(-20) + 0 = 37$

Ex Upper Triangular Matrices:

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \\ 6 \end{vmatrix} = 1 \cdot \begin{vmatrix} 4 & 5 \\ 6 \end{vmatrix} = 1 \cdot 4 \cdot 6$$

$$A_{ij} = 0 \text{ if } i < j \Rightarrow \det A = \prod_i A_{ii}$$

Products

$$\boxed{\det AB = \det A \cdot \det B}$$

Very useful!
See book for proof.

Cor If A^{-1} exists, then

$$\det A^{-1} = \frac{1}{\det A}$$

Col Props of Determinant

Think of matrix as n column vectors:

$$A = \left(\begin{array}{c|c|c|c} A_{11} & A_{21} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{array} \right) = (A_1 | A_2 | \dots | A_n)$$

$\underbrace{\hspace{1cm}}_{A_1} \quad \underbrace{\hspace{1cm}}_{A_2} \quad \underbrace{\hspace{1cm}}_{A_n}$

The following follow easily from the definition.

(antisymmetry) Swap any 2 cols, get $-$ sign

$$\det(A_1, A_2, \dots, A_n) = -\det(A_2, A_1, \dots, A_n)$$

(multilinearity) $\det$ is linear with respect to each column:

$$\det(kA_1 + k'B, A_2, \dots, A_n) = k \det(A_1, A_2, \dots, A_n) + k' \det(B, A_2, \dots, A_n)$$

Consequences

1. Scale a column by k , scale $\det$ by k .
2. If any 2 columns equal, $\det A = 0$ by antisymmetry:
$$\det(A_1, A_1, A_3, \dots, A_n) = -\det(A_1, A_1, A_3, \dots, A_n)$$
3. If any 2 columns proportional, $\det A = 0$ by 1+2.
4. Can add multiples of 1 column to another, $\det$ is invariant by multilinearity and 3.

Comment: All above works just as well for rows.

(b/c $\det A^T = \det A$ will discuss soon)
 ↑
 transpose ←

Cramer's rule: inverting A using det

Approach to solving $A\bar{x} = \bar{y}$ for x explicitly.

Let

$$D = \det(A) = \begin{vmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & \dots & \dots & A_{nn} \end{vmatrix}$$

To find x_1 , multiply 1st col by x_1 :

$$x_1 D = \begin{vmatrix} x_1 A_{11} & A_{12} & \dots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 A_{n1} & \dots & \dots & A_{nn} \end{vmatrix}$$

Add multiples of other cols to col 1. Det unchanged.

$$x_1 D = \begin{vmatrix} x_1 A_{11} + x_2 A_{12} + \dots + x_n A_{1n} & A_{12} & \dots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \underbrace{x_1 A_{n1} + x_2 A_{n2} + \dots + x_n A_{nn}}_{\text{column vector } \bar{y}} & A_{n2} & \dots & A_{nn} \end{vmatrix}$$

So long as $D \neq 0$:

$$x_1 = \frac{1}{D} |y, A_2, \dots, A_n| \quad \text{column notation for matrix}$$

Generalizing to x_i :

$$x_i = \frac{1}{D} |A_1, \dots, \overset{\text{i}^{\text{th}} \text{ column}}{y}, \dots, A_n|$$

Cramer's Rule
column form

Thm

The following are equivalent:

- i) $\det A \neq 0$
- ii) $A\bar{x} = \bar{y}$ has a unique solution $\bar{x}$ for any $\bar{y}$
- iii) A^{-1} exists (and is unique)
- iv) the columns of A form a basis

Comparing Cramer's rule to $\bar{x} = A^{-1}\bar{y}$ and using expansion by minors:

$$(A^{-1})_{ij} = \frac{(-1)^{i+j}}{D} M_{ji}$$

Cramer's rule, cofactor form

↑
notice order of indices

Why? Choose $\bar{y} = \hat{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j^{\text{th}} \text{ row}$

$$x = A^{-1}y \Rightarrow x_i = \sum_k (A^{-1})_{ik} y_k = (A^{-1})_{ij}$$

Cramer's rule in column form:

$$x_i = \frac{1}{D} \begin{vmatrix} A_{11} & \dots & 0 & \dots & A_{1n} \\ \vdots & & \vdots & & \vdots \\ A_{j1} & \dots & 1 & \dots & A_{jn} \\ \vdots & & \vdots & & \vdots \\ A_{n1} & \dots & 0 & \dots & A_{nn} \end{vmatrix}$$

← $j^{\text{th}} \text{ row}$

↑
 $i^{\text{th}} \text{ col}$

Expand i^{th} col by minors. Only non-zero term is j^{th} :

$$= \frac{1}{D} (-1)^{i+j} M_{ji}$$

□

Efficiency of Cramer's Rule

- Cramer's rule is closed form for inverse but not efficient in practice

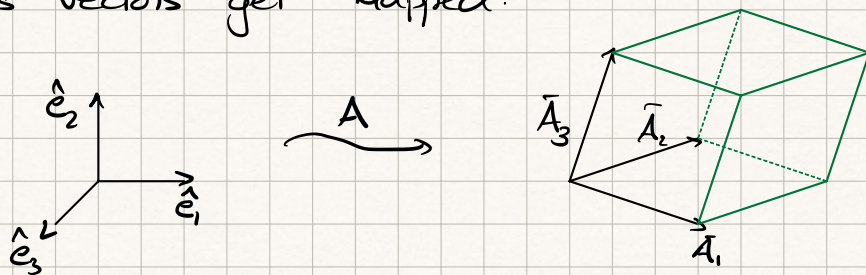
Q? How many operations to evaluate Cramer's rule for even a single x ?

Two alts: $O(n!)$

- Gaussian elimination (next discussion) much faster
- /HW

Geometric Interpretation of Determinant

Column vectors of $A = (A_1 | A_2 | \dots | A_n)$ specify where basis vectors get mapped:



$\det A =$ oriented n -volume of parallelepiped $\bar{A}_1, \bar{A}_2, \dots, \bar{A}_n$
 sign + if orientation same as $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n$

Ex $A = (3)$ $\det A = 3$
 $\hookrightarrow$ volume = length = 3

Ex $A = (-3)$ $\det A = -3$
 length = 3 but reversed orientation

Ex $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\det A = 1$

area = 1 $\bar{A}_1, \bar{A}_2$ still right-handed

Ex $n=3$ $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\det A = 0$

$\hookrightarrow$ no volume b/c all 3 $\bar{A}$ lie in a plane.