

Linear Maps $\rightarrow$ Matrices

Linear maps/transformations take vectors in one vector space to another linearly.

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$\nearrow$
Real n -dimensional space

T eats a vector and gives another vector.

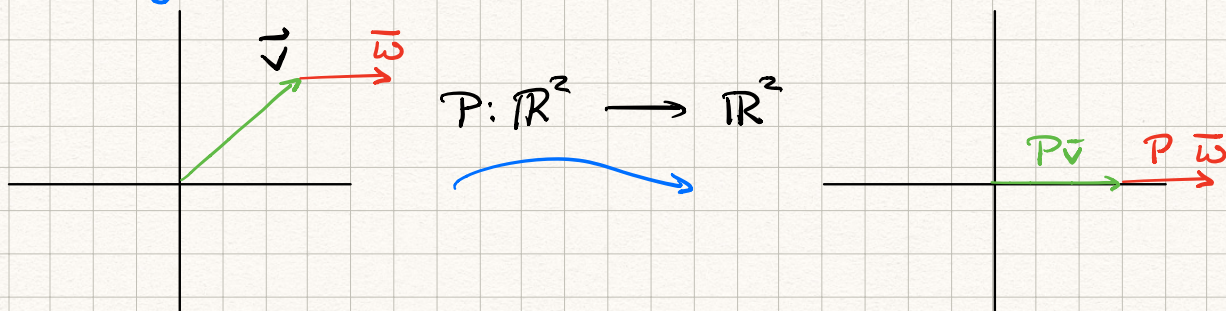
$$T \vec{v} = \vec{u}$$

$\nearrow$ in $\mathbb{R}^n$ $\nwarrow$ in $\mathbb{R}^m$

Linearity: $T(a\vec{v} + b\vec{w}) = aT\vec{v} + bT\vec{w}$

- Multiplication by scalars commutes with T
- T distributes over sums

Ex Projection to x -axis



Some linear transformations: rotations, projections, shears, scales

- physical basis-independent view.

To do calculations, introduce bases for vectors, then linear transformation represented by $m \times n$ matrix.

$$\begin{aligned} \vec{v} = \sum_i v_i \hat{e}_i &\iff T\vec{v} = \sum_i v_i \underbrace{T\hat{e}_i}_{\text{some vector, write in basis } \hat{e}_j'} \\ &= \sum_i v_i \sum_j T_{ji} \hat{e}_j' = \sum_j \left(\sum_i T_{ji} v_i \right) \hat{e}_j' \end{aligned}$$

$m \times n$ matrix elements.

Matrices

→ rectangular arrays of numbers (matrix elements)

$$A\vec{v} = \vec{w} \quad \begin{matrix} \text{in basis} \\ \downarrow \end{matrix} \Leftrightarrow \begin{matrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} & = & \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ \text{2x2 matrix} & \text{2d col vector} & & \text{2d column vector} \end{matrix}$$

Q? How many equations?

$$\Leftrightarrow \begin{aligned} a_{11}v_1 + a_{12}v_2 &= w_1 \\ a_{21}v_1 + a_{22}v_2 &= w_2 \end{aligned}$$

$m \times n$ matrix has m rows and n columns.

A_{ij} element in i^{th} row, j^{th} column.

Warning: Math usually uses 1-indexing. Python uses 0-indexing.

Matrix Operations

Addition/subtraction: by element

$$(A + B)_{ij} = A_{ij} + B_{ij}$$

Q? $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = ?$ Not defined! Must have same shape!

– commutative, associative, zero matrix $= \begin{pmatrix} 0 & 0 \\ 0 & \ddots \end{pmatrix} = \mathbf{0}$
 $A + \mathbf{0} = \mathbf{0} + A = A$

Multiplication by scalars: $(kA)_{ij} = kA_{ij}$ (multiply all elements)

– distributive: $k(A+B) = kA + kB$

∴ Set of $m \times n$ matrices is itself a vector space

Matrix multiplication: NOT element by element

$$C = AB \quad \Leftrightarrow \quad C_{ij} = \sum_k A_{ik} B_{kj}$$

Need shapes to be compatible:

$$(n \times m) \cdot (m \times p) \rightarrow (n \times p) \text{ matrix}$$

Special case: matrix · column vector:

$$A\bar{x} = \bar{y} \iff \sum_j A_{ij} x_j = y_i$$

$(n \times m) \cdot (m \times 1) \rightarrow (n \times 1)$

Ex Projection to x-axis.

Standard basis: $\hat{x}, \hat{y}$

$$\bar{v} = v_x \hat{x} + v_y \hat{y} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

$$P\bar{v} = v_x \hat{x}$$

How to get matrix of P ?

Act P on basis vectors. 1st column = $P\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, 2nd = $P\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, ...

$$P = (P\hat{x} | P\hat{y}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Systems of Linear eqns

- Matrices give compact way to write systems of linear equations

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 &= y_1 \\ A_{21}x_1 + A_{22}x_2 &= y_2 \\ A_{31}x_1 + A_{32}x_2 &= y_3 \end{aligned} \iff \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\iff A\bar{x} = \bar{y}$$

3 equations, 2 unknowns $\iff 3 \times 2$ matrix A

Square Matrices

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & & & \\ \vdots & & & \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}$$

$\xleftarrow{\quad n \quad} \xrightarrow{\quad}$

- Square $n \times n$ matrices are most important special case.

$A\vec{v} = \vec{w}$ - Represents linear map from space back to itself:

$$A: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$A\vec{x} = \vec{b}$ - n equations in n unknowns x_i :

A - physical observable in quantum mechanics

Basics

- AB and BA both defined. Q? why?

BUT not commutative!

$$AB \neq BA \quad \text{in general}$$

- Commutator: $[A, B] = AB - BA$ measures deviation from commutativity
(see HW)

- Diagonal matrix:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 3 & 0 & \dots \\ 0 & 0 & 0 & \ddots & \end{pmatrix} = \begin{pmatrix} 2 & & & \\ & 1 & & \\ & & 3 & \\ & & & \ddots \end{pmatrix}$$

all off diagonal matrix elts = 0

often don't write 0's explicitly.

- Identity matrix: $\mathbb{I} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$ ← different $\mathbb{I}$ for every n but usually clear from context

- $1\bar{v} = \bar{v}$ identity map for any $\bar{v}$

- $A \mathbb{1} = \mathbb{1} A = A$ identity for matrix multiplication for any A

- elements given by Kronecker delta!

$$\mathbb{1}_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

Matrix Inverse

If $AB = BA = \mathbb{1}$ then B is inverse of A . Write $B = A^{-1}$.

Ex $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $B = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $Q?$ $AB = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $BA = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$$\therefore B = A^{-1} \quad (\text{and } A^{-1} = B)$$

- If inverse exists, then it is **unique**.

Why? Suppose B, C both inverse to A . Then

$$\begin{aligned} CAB &= C(AB) = C1 = C \Rightarrow B = C \\ &= (CA)B = 1B = B \end{aligned}$$

- If A has no inverse, it is called **singular**.

Ex $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

Q? $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

But if A^{-1} existed, then $\underset{1B}{A^{-1}AB = A^{-1}O = O} \Rightarrow B=0 \Rightarrow \text{X}$

$\therefore A$ is singular.

Application: Solving systems of equations

$$\begin{array}{l} A_{11}x_1 + \dots + A_{1n}x_n = b_1 \\ \vdots \\ A_{n1}x_1 + \dots + A_{nn}x_n = b_n \end{array} \Leftrightarrow A\bar{x} = \bar{b} \quad \text{matrix notation}$$

coefs $\nearrow$ $\nwarrow$ unknowns $\nearrow$ constants

$$A^{-1}A\bar{x} = A^{-1}\bar{b} \quad \text{if } A^{-1} \text{ exists}$$
$$\boxed{\bar{x} = A^{-1}\bar{b}}$$

- Solution $\bar{x}$ is unique if A is nonsingular.

Inverse of Product

If A, B both nonsingular, then $(AB)^{-1} = B^{-1}A^{-1}$

$\uparrow$
reverse order

Why? Just check:

$$\left[\begin{array}{l} (B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}B = I \\ \text{and similarly for } (AB)(B^{-1}A^{-1}). \end{array} \right]$$