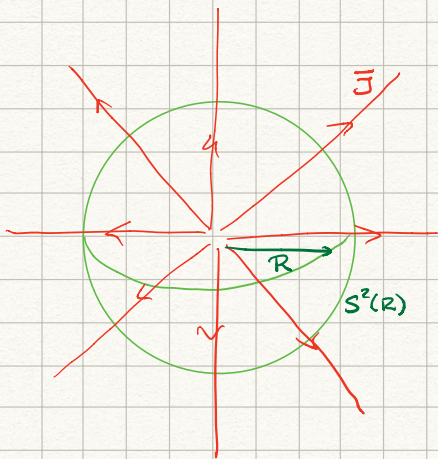


The Dirac Delta Function

Griffiths Electrodynamics 1.5

A Paradox?



Consider a radially outward field:

$$\vec{J} = \frac{1}{r^2} \hat{r}$$

Clearly has flux through spherical surface:

$$\oint_{S^2(R)} \vec{J} \cdot d\vec{\sigma} \stackrel{Q?}{=} \oint_{S^2(R)} \frac{1}{R^2} \hat{r} \cdot \hat{r} dA = \frac{1}{R^2} \oint dA = \frac{4\pi R^2}{R^2} = 4\pi$$

So must have divergence:

$$4\pi = \oint_{S^2(R)} \vec{J} \cdot d\vec{\sigma} = \int_{\text{inside } S^2(R)} \nabla \cdot \vec{J} d\tau$$

But: (calculated by going to Cartesian in div lecture)

$$\begin{aligned} \nabla \cdot \left(\frac{1}{r^2} \hat{r} \right) &\stackrel{Q?}{=} \nabla \cdot \left(\frac{1}{r^3} \vec{r} \right) = \nabla \left(\frac{1}{r^3} \right) \cdot \vec{r} + \frac{1}{r^3} \nabla \cdot \vec{r} \\ &= -3r^{-4} \underbrace{(\nabla r)}_{\hat{r}} \cdot \vec{r} + \frac{3}{r^3} \\ &= \frac{-3+3}{r^3} = 0 \Rightarrow \int \nabla \cdot \vec{J} d\tau = 0 !!! \end{aligned}$$

Where is the problem?

- Divergence theorem is correct.

- At $r=0$, $\vec{J}$ blows up and $\nabla \cdot \vec{J}$ is %

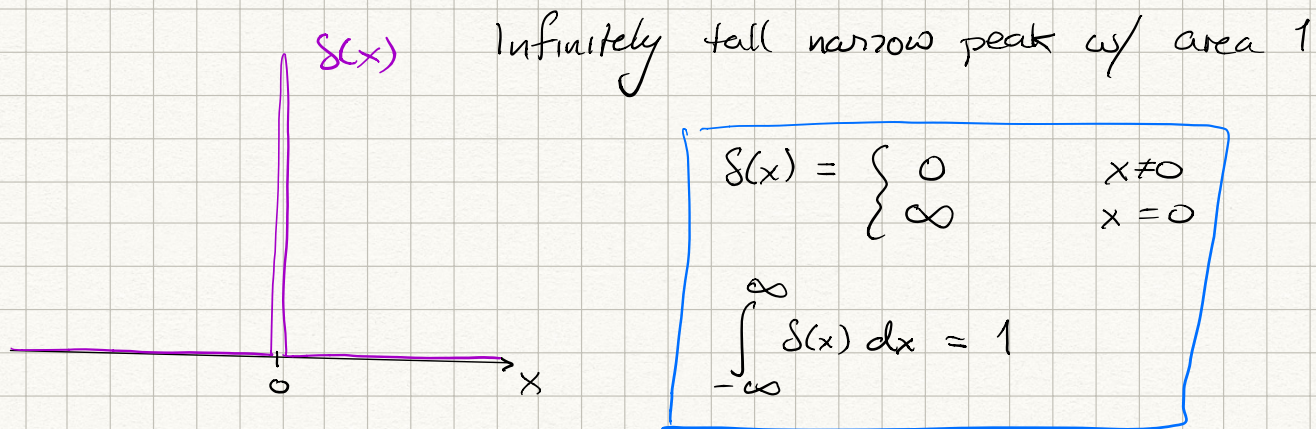
Notice: $\int_{\text{inside } S^2(R)} \nabla \cdot \vec{J} d\tau = 4\pi$ is independent of radius R of sphere.

All contribution must be from origin.

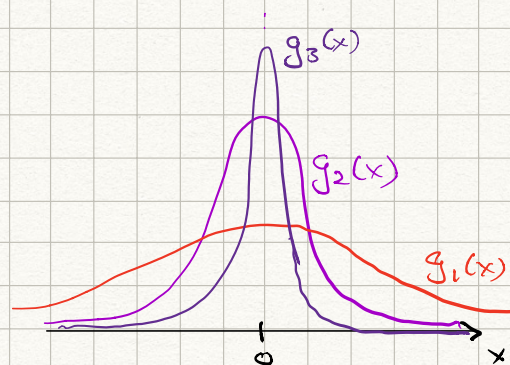
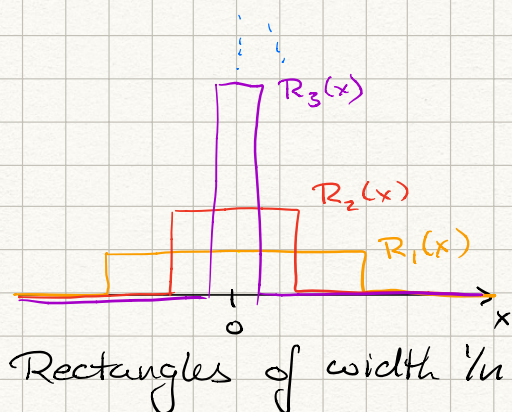
→ Point-like region with finite contribution to integral.

$$\boxed{\nabla \cdot \frac{\hat{r}}{r^2} = 4\pi \delta^{(3)}(\vec{r})} \quad \leftarrow \text{3D Dirac Delta Function}$$

1D Dirac Delta Function



- Mathematically, $\delta(x)$ is a generalized function or distribution
- Intended to be integrated (Excellent ref: Lighthill '59)



Normalized Gaussian of variance $1/n$

$$g_n(x) = \sqrt{\frac{n}{2\pi}} e^{-nx^2/2}$$

- $\delta(x)$ is limit after integration of sequence of normalized bumps

$$\int f(x) \delta(x) dx \equiv \lim_{n \rightarrow \infty} \int f(x) g_n(x) dx$$

How to use it

- for ordinary function (eg. continuous):

$$f(x) \delta(x-a) = f(a) \delta(x-a) \quad \leftarrow \text{shifted bump at } a$$

Why? B/c $\delta(x-a) = 0$ for $x \neq a$ so product is 0 anyway except at a

$$\textcircled{1} \quad \int_{-\infty}^{\infty} f(x) \delta(x-a) dx = f(a) \int_{-\infty}^{\infty} \delta(x-a) dx = f(a)$$

$\delta(x-a)$ picks out value of $f(x)$ at a .

Ex $\int_{-\infty}^{\infty} (x^3 - 3 \sin x) \delta(x-1) dx = 1 - 3 \sin 1$

$$\int_{-\infty}^0 (x^3 - 3 \sin x) \delta(x-1) dx = 0 \quad \text{Doesn't include bump in domain!}$$

- The distributions (expressions using δ -functions) D_1 & D_2 are equal if

$$\int_{-\infty}^{\infty} D_1(x) f(x) dx = \int_{-\infty}^{\infty} D_2(x) f(x) dx$$

for all ordinary "test" functions $f(x)$.

- D_1, D_2 behave the same way in every integral.

Ex Show $\delta(kx) = \frac{1}{|k|} \delta(x)$

pf for arbitrary test function $f(x)$:

$$\int_{-\infty}^{\infty} f(x) \delta(kx) dx = \int_{-k\infty}^{k\infty} f(u/k) \delta(u) \frac{du}{k} = \pm \int_{-\infty}^{\infty} f(u) \frac{\delta(u)}{k} du = \pm \frac{f(0)}{k} = \frac{f(0)}{|k|}$$

$$u = kx \quad du = k dx$$

$\uparrow$
-ve if $k < 0$
b/c limits reverse

$$\text{That's same as: } \int_{-\infty}^{\infty} f(x) \frac{\delta(x)}{|k|} dx = \frac{f(0)}{|k|}$$

□

General trick to evaluating f-function expressions is to use integral tricks to make the expression look like integrating prop ①.

Props $\int f(x) g'(x) dx = -f'(0)$

⋮

See book and HW.

3D Dirac Delta Function

$$\delta^{(3)}(\vec{r}) = \delta(x)\delta(y)\delta(z) = \begin{cases} 0 & \vec{r} \neq \text{origin} \\ \infty & \vec{r} = \text{origin} \end{cases}$$

$$\int_{\substack{\text{all space} \\ \mathbb{R}^3}} \delta^{(3)}(\vec{r}) d\tau = \int \delta(x) dx \int \delta(y) dy \int \delta(z) dz = 1$$

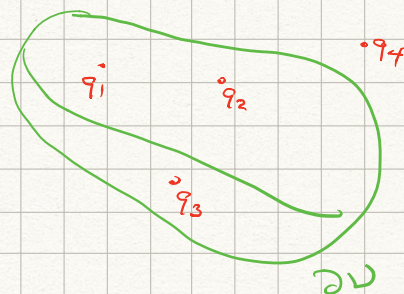
- Represents mass density of point mass. Infinite spike of mass 1.

Ex
$$\int_{\mathbb{R}^3} \delta^{(3)}(\vec{r}-\vec{a}) f(\vec{r}) d\tau = f(\vec{a})$$

Ex In electrostatics, Gauss' law:

$$\oint_{\partial V} \vec{E} \cdot d\vec{\sigma} = \frac{\text{Charge in } V}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho(\vec{r}) d^3r$$

↑
charge density



Can represent set of point charges q_i at positions $\vec{R}_i$:

$$\rho(\vec{r}) = \sum_i q_i \delta^{(3)}(\vec{r} - \vec{R}_i)$$

Get:
$$\oint_{\partial V} \vec{E} \cdot d\vec{\sigma} = \frac{1}{\epsilon_0} \int_V \sum_i q_i \delta^{(3)}(\vec{r} - \vec{R}_i) d^3r = \sum_{\substack{i \\ R_i \in V}} q_i / \epsilon_0$$

Since Gauss' law holds for any V :

$$\oint_{\partial V} \vec{E} \cdot d\vec{\sigma} = \int_V \rho / \epsilon_0 d\tau$$

||

$$\int_V \vec{\nabla} \cdot \vec{E} d\tau \Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \rho / \epsilon_0}$$

So Gauss' law $\Leftrightarrow$ Maxwell's equation in differential form so long as we recall each side is a distribution (can contain δ -functions)