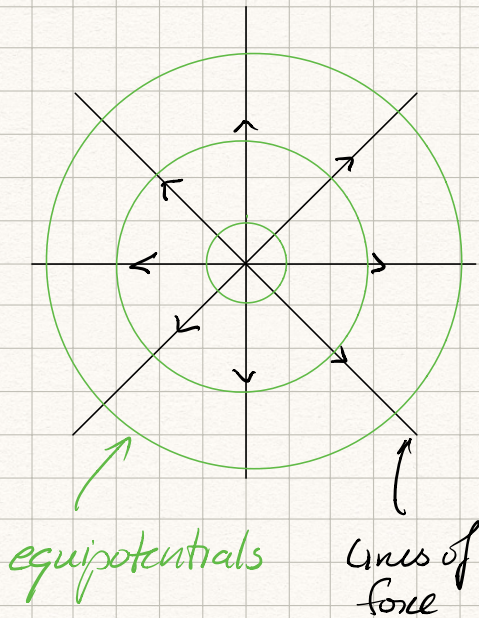


Scalar Potentials



A scalar field whose gradient is a force

$$\vec{F} = -\nabla\phi$$

is called a scalar potential.

- simpler to work w/ 1 component scalar than 3 component vector
- encodes physics of conservative forces

Ex Coulomb potential from point charge Q

$$\phi(r) = k \frac{Q}{r}$$

Force on test charge q : $\vec{F} = -q \nabla\phi = k \frac{Qq}{r^2} \hat{r}$

Electric field: $\vec{E} = \frac{\vec{F}}{q} = -\nabla\phi$

Ex Gravity near Earth

attractive for +ve m, M
compare to Coulomb

$z \gg R_E$

Newton: $\phi(r) = -\frac{GMm}{r}$

Near Earth: $z \ll R_E$

- Easier to expand ϕ than $\vec{F}$

$$\phi(R_E + z) = -\frac{GMm}{R_E + z} = -\frac{GMm}{R_E} \frac{1}{1 + z/R_E}$$

$$= -\frac{GMm}{R_E} (1 - z/R_E + \dots)$$

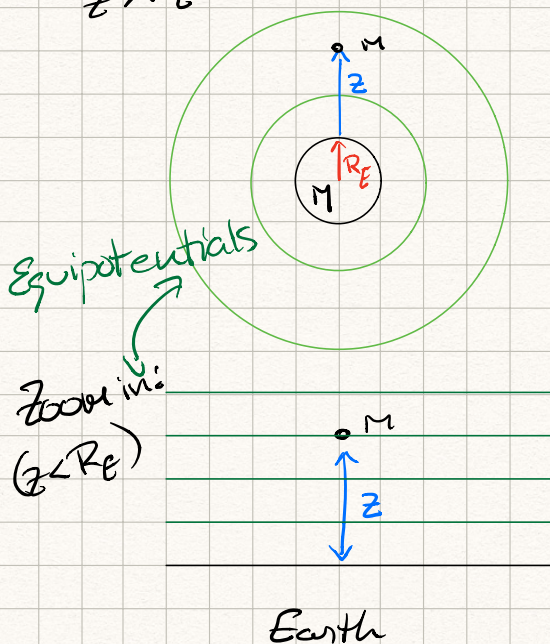
$$\phi = -\frac{GMm}{R_E} + \frac{GM}{R_E^2} mz + \dots$$

$$= mgz + \text{const} + \dots$$

2 familiar!

doesn't matter for $\vec{F}$

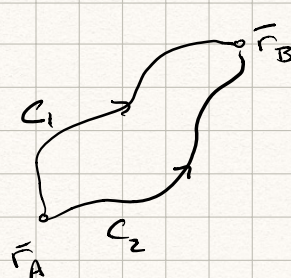
$$\vec{F} = -\nabla\phi = -mg\hat{z} + \dots$$



Note: Scalar ϕ is not unique:

$$\boxed{\phi \rightarrow \phi + \text{constant} \text{ leaves } -\nabla\phi \text{ invariant}}$$

A force $\vec{F}$ is **conservative** if the work done against $\vec{F}$ moving between $\vec{r}_A$ and $\vec{r}_B$ is independent of the path:



$$W_{C_1} = - \int_{\vec{r}_A, C_1}^{\vec{r}_B} \vec{F} \cdot d\vec{r} = - \int_{\vec{r}_A, C_2}^{\vec{r}_B} \vec{F} \cdot d\vec{r} = W_{C_2}$$

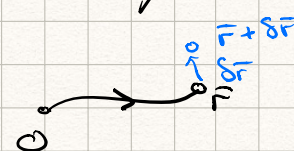
$$\boxed{\vec{F} \text{ is conservative} \Leftrightarrow \exists \text{ scalar potential } \phi \text{ st } \vec{F} = -\nabla\phi}$$

pf ($\Leftarrow$) Suppose $\vec{F} = -\nabla\phi$ then

$$W_C = + \int_{\vec{r}_A, C}^{\vec{r}_B} \nabla\phi \cdot d\vec{r} = \phi(\vec{r}_B) - \phi(\vec{r}_A) \quad \text{by Gradient theorem}$$

So independent of C .

($\Rightarrow$) If $\vec{F}$ is conservative, pick a reference point O and define:

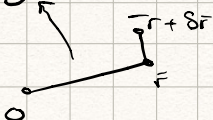
$$\phi(\vec{r}) = - \int_O^{\vec{r}} \vec{F} \cdot d\vec{r}$$


This is well-defined b/c choice of path doesn't matter.

What is $\nabla\phi$?

$$\phi(\vec{r} + \delta\vec{r}) - \phi(\vec{r}) = \int_O^{\vec{r} + \delta\vec{r}} -\vec{F} \cdot d\vec{r} - \int_O^{\vec{r}} -\vec{F} \cdot d\vec{r}$$

Choose paths:



$$= \int_{\vec{r}, \text{straight line}}^{\vec{r} + \delta\vec{r}} -\vec{F} \cdot d\vec{r} \approx - \underbrace{\vec{F}(\vec{r})}_{\nabla\phi} \cdot \delta\vec{r}$$

□

More generally, the following are equivalent:

- 1) $\vec{F}$ is conservative
- 2) $\exists$ scalar potential ϕ st $\vec{F} = -\vec{\nabla}\phi$
- 3) $\vec{\nabla} \times \vec{F} = 0$ ($\vec{F}$ is irrotational)
- 4) $\oint_C \vec{F} \cdot d\vec{\ell} = 0$ any closed loop C

Easy to show equivalence. Already showed $1 \Leftrightarrow 2$

$(2 \Rightarrow 3)$ If $\vec{F} = -\vec{\nabla}\phi$, then

Taking a curl: $\vec{\nabla} \times \vec{F} = -\vec{\nabla} \times (\vec{\nabla}\phi) = 0$ (2nd derivative lecture)

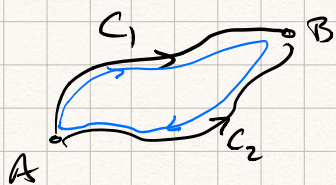
$(3 \Rightarrow 4)$ If $\vec{\nabla} \times \vec{F} = 0$, then for any closed loop C ,

$$\oint_{C=\partial\Sigma} \vec{F} \cdot d\vec{\ell} = \int_{\Sigma} \vec{\nabla} \times \vec{F} \cdot d\vec{\sigma} = 0$$

$\uparrow$ surface w/ boundary C

$(4 \Rightarrow 1)$ If $\oint_C \vec{F} \cdot d\vec{\ell} = 0$ for any closed loop C ,

then $W = -\int_A^B \vec{F} \cdot d\vec{\ell}$ is independent of path:



$$\underbrace{\int_{C_1} \vec{F} \cdot d\vec{\ell}}_{W_1} - \underbrace{\int_{C_2} \vec{F} \cdot d\vec{\ell}}_{W_2} = \oint_{C_1 - C_2} \vec{F} \cdot d\vec{\ell} = 0$$

$\therefore W_1 = W_2$ is indep of C .

Vector Potentials

A vector field $\vec{A}$ whose curl is another field $\vec{B}$ is a vector potential:

$$\vec{B} = \nabla \times \vec{A}$$

- Same number of components as $\vec{B}$, so why bother?
- $\vec{B}$ is automatically solenoidal:

$$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$$

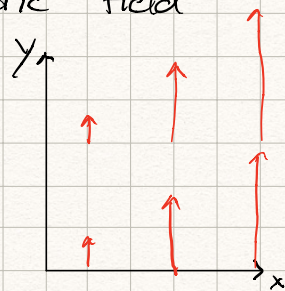
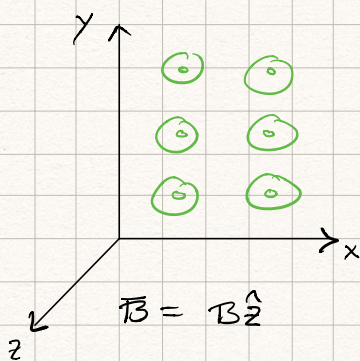
So using potential $\vec{A}$ solves one of Maxwell's equations by construction.

(uniqueness) $\vec{A}$ is not unique:

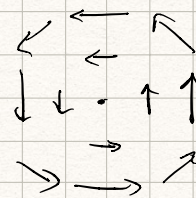
$$\vec{A} \rightarrow \vec{A} + \nabla \chi \quad \text{leaves } \vec{B} = \nabla \times (\vec{A} + \nabla \chi) = \nabla \times \vec{A} \text{ invariant}$$

↑ called a gauge transformation

Ex Uniform magnetic field



(Landau gauge)



$$\vec{A} = \frac{B}{2}(x\hat{y} - y\hat{x}) = \frac{B}{2} r \hat{\phi}$$

(Symmetric gauge)

Putting Potentials Together

$$\vec{F} \text{ conservative} \Leftrightarrow \vec{\nabla} \times \vec{F} = 0 \Leftrightarrow \exists \phi \text{ st } \vec{F} = -\vec{\nabla} \phi$$

irrotational

$$\vec{F} \text{ solenoidal} \Leftrightarrow \vec{\nabla} \cdot \vec{F} = 0 \Leftrightarrow \exists \vec{A} \text{ st } \vec{F} = \vec{\nabla} \times \vec{A}$$

In general, can decompose any $\vec{F}$ as

$$\vec{F} = \underbrace{-\vec{\nabla} \phi}_{\text{longitudinal}} + \underbrace{\vec{\nabla} \times \vec{A}}_{\text{transverse parts of } \vec{F}}$$

longitudinal & transverse parts of $\vec{F}$.

where ϕ determines $\vec{\nabla} \cdot \vec{F}$ and $\vec{A}$ determines $\vec{\nabla} \times \vec{F}$.

Helmholtz Theorem $\vec{F}$ is determined on a volume V by

$$\vec{\nabla} \cdot \vec{F}, \vec{\nabla} \times \vec{F} \text{ and } \vec{F} \cdot \hat{n} \text{ on } \partial V.$$