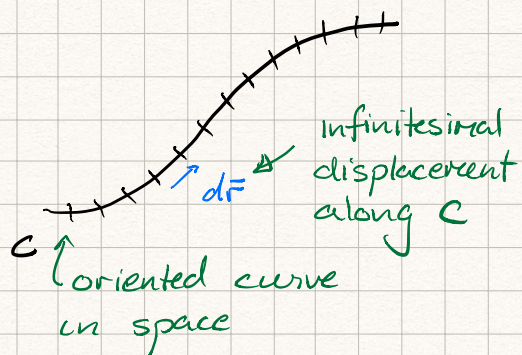


Vector Calculus: Integration

Line Integrals (domain of integration = curve in space)



$$\int_C \phi d\vec{r}$$

Q? vector or scalar?
vector

$$\int_C \vec{F} \cdot d\vec{r}$$

scalar

Work done by $\vec{F}$
on particle moving
along C

$$\int_C \vec{\nabla} \times d\vec{r}$$

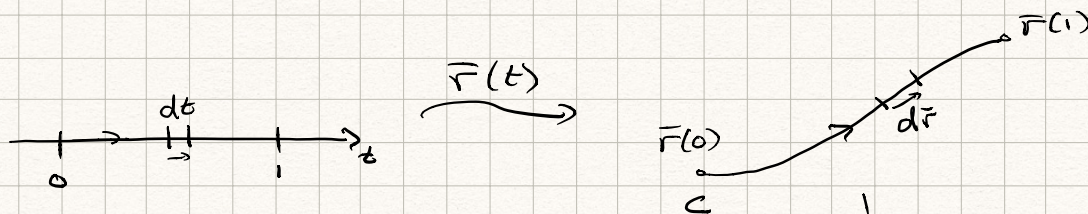
vector

Idea: Divide curve into lots of little pieces and add up integrand.

$\int$ is just a weird Σ for sum.

To explicitly evaluate need to parametrize curve C :

Find $\vec{r}(t)$ which traces curve C as t varies.



This reduces line integral on C to standard $\int_0^1 \dots dt$.

Often can do a lot w/out parametrizing using integral theorems/geometry.

Today: some explicit integrals in Cartesian coordinates:

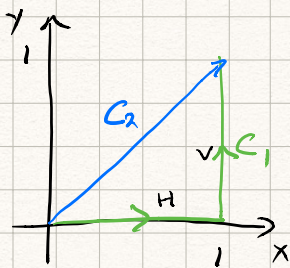
$$d\vec{r} = dx \hat{x} + dy \hat{y} + dz \hat{z}$$

Above can be written in components. Eg:

$$\int_C \phi d\vec{r} = \hat{x} \int_C \phi(x,y,z) dx + \hat{y} \int_C \phi(x,y,z) dy + \hat{z} \int_C \phi(x,y,z) dz$$

unit vectors come out of $\int$ b/c they are constants

Ex



$$\varphi(x,y) = xy$$

$$\overline{I}_C = \int_C \varphi d\vec{r}$$

$$\overline{I}_{C_1} = \int_C \varphi d\vec{r} = \int_{\text{Horiz}} \varphi d\vec{r} + \int_{\text{vert}} \varphi d\vec{r}$$

$x \in [0,1] \quad y=0 \quad d\vec{r} = \hat{x}dx$
 $x=1 \quad y \in [0,1] \quad d\vec{r} = \hat{y}dy$

$$= \int_0^1 \varphi(x,0) \hat{x} dx + \int_0^1 \varphi(1,y) \hat{y} dy$$

$$= 0 + \hat{y} \int_0^1 y dy = \hat{y} \left. \frac{y^2}{2} \right|_0^1 = \frac{1}{2} \hat{y}$$

$$\overline{I}_{C_2} = \int_{C_2} xy d\vec{r} = \int_0^1 x^2 \cdot (\hat{x} dx + \hat{y} dx) = (\hat{x} + \hat{y}) \int_0^1 x^2 dx = (\hat{x} + \hat{y}) \left. \frac{x^3}{3} \right|_0^1$$

$\begin{matrix} C_2 \text{ Use } x \text{ as param:} \\ y=x \in [0,1] \\ dy=dx \end{matrix} \quad = \frac{1}{3}(\hat{x} + \hat{y})$

Ex

Work integral on same two paths.

$$W_C = \int_C \vec{F} \cdot d\vec{r} \quad \vec{F} = -y\hat{x} + x\hat{y}$$

On C_1 :

$$\begin{aligned} W_{C_1} &= \int_0^1 dx \hat{x} \cdot \vec{F}(x,0) + \int_0^1 dy \hat{y} \cdot \vec{F}(1,y) \\ &= \int_0^1 dx 0 + \int_0^1 dy 1 = 1 \end{aligned}$$

On C_2 :

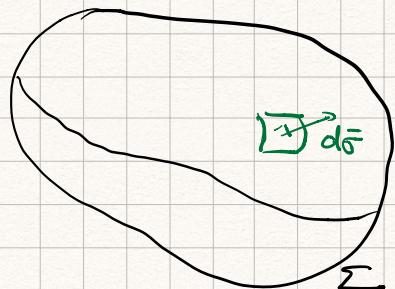
$$\begin{aligned} W_{C_2} &= \int_0^1 \vec{F}(x,x) \cdot d\vec{r} = \int_0^1 (-x\hat{x} + x\hat{y}) (\hat{x} dx + \hat{y} dx) \\ &= \int_0^1 (-x + x) dx = 0 \end{aligned}$$

- Line integrals depend on path C in general.

Surface Integrals

(domain = 2D surface in space)

Closed surface



$$\int_{\Sigma} \vec{v} \, d\vec{\sigma}$$

vector

$$\int_{\Sigma} \vec{v} \cdot d\vec{\sigma}$$

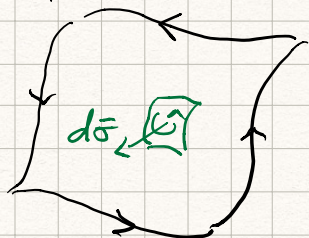
scalar

flux of $\vec{v}$ through Σ

$$\int_{\Sigma} \vec{v} \times d\vec{\sigma}$$

vector

Open surface



$$d\vec{\sigma} = \hat{n} \, dA$$

← area of element

↑ direction of normal

↑ element of oriented surface

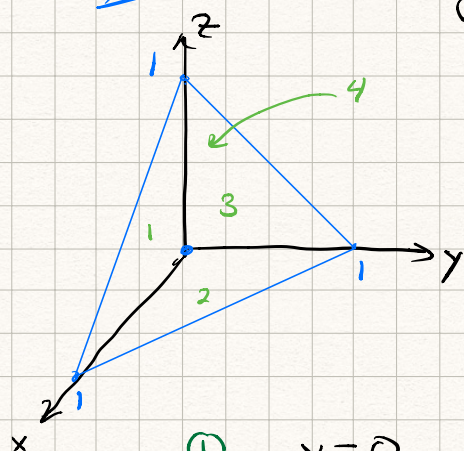
Orientation of $\hat{n}$:

- If Σ is closed, conventionally take $\hat{n}$ outward.
 - If Σ is open (has a boundary), need to specify
- Choice of $\hat{n}$ induces choice of orientation on $\partial\Sigma$

Parameterization of Σ requires 2 parameters. "boundary of"

- Painful → look for geometric simplifications first

Ex Flux out of tetrahedron?



$$\vec{E} = (x+1)\hat{x} + y\hat{y} - z\hat{z}$$

$$\oint \vec{E} \cdot d\vec{\sigma} = \sum_{4 \text{ faces}} \int_{\text{face}} \vec{E} \cdot d\vec{\sigma}$$

Tetrahedron
Put 0 to reveal surface is closed.

① $y=0$ $\hat{n} = -\hat{y}$ $\vec{E} \cdot \hat{n} = -y = 0 \Rightarrow \text{No flux.}$

② $z=0$ $\hat{n} = -\hat{z}$ $\vec{E} \cdot \hat{n} = z = 0 \Rightarrow \text{No flux.}$

③ $x=0$ $\hat{n} = -\hat{x}$ $\vec{E} \cdot \hat{n} = -1$

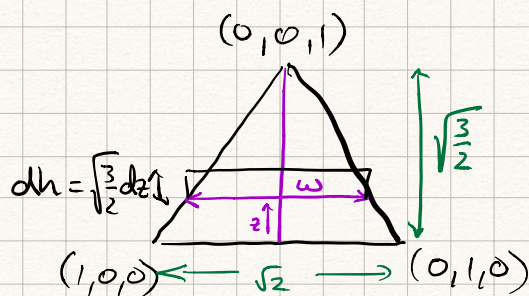
$$\int_3 \vec{E} \cdot \hat{n} dA = \int_3 -1 dA = -(\text{Area}) = -\frac{1}{2}$$

④ $x+y+z=1$ $\hat{n} = \frac{1}{\sqrt{3}}(\hat{x} + \hat{y} + \hat{z})$

$$\vec{E} \cdot \hat{n} = \frac{1}{\sqrt{3}}[x+1 + y - z] = \frac{2}{\sqrt{3}}[1-z]$$

$$\int_4 \vec{E} \cdot \hat{n} dA = \frac{2}{\sqrt{3}} \int_4 (1-z) dA$$

To evaluate last integral need dA on $\Delta 4$



z goes from 0 to 1

at z , width is $\sqrt{2}(1-z)$

height is $\sqrt{\frac{3}{2}}$

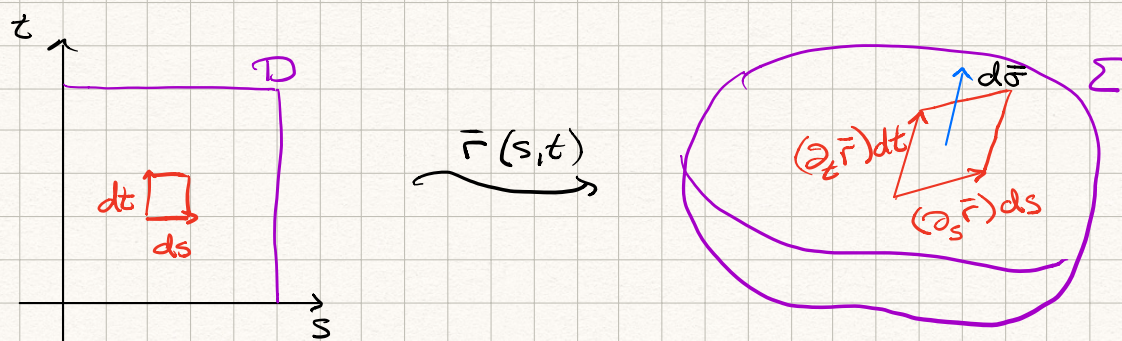
So $dA = \sqrt{2}(1-z) \sqrt{\frac{3}{2}} dz$

$$\frac{2}{\sqrt{3}} \int_4 (1-z) dA = 2 \int_0^1 (1-z)^2 dz = 2 \left(-\frac{1}{3}(1-z)^3 \right) \Big|_0^1 = \frac{2}{3}$$

Bottom Line: $\oint_{\text{tet}} \vec{E} \cdot d\vec{\sigma} = -\frac{1}{2} + \frac{2}{3} = \frac{1}{6}$

General Case (not necessary for class but good to know):

If $\vec{r}(s,t)$ parameterizes surface Σ :



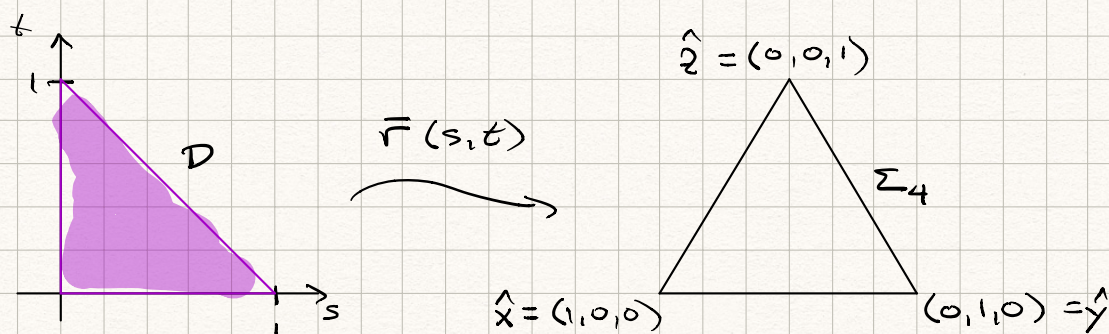
then oriented surface element $d\vec{S} = \hat{n}(\text{area of } \square)$
 $= (\partial_s \vec{r} ds) \times (\partial_t \vec{r} dt)$

$$d\vec{S} = (\partial_s \vec{r} \times \partial_t \vec{r}) ds dt$$

Ex Alternate param of face (4):

$$\vec{r} = \hat{x} + s(\hat{y} - \hat{x}) + t(\hat{z} - \hat{x})$$

$$0 \leq s \quad 0 \leq t \quad s+t \leq 1$$



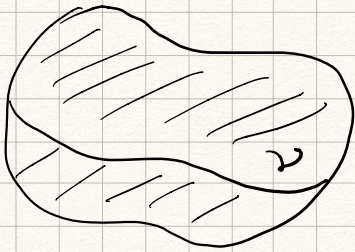
$$d\vec{S} = \partial_s \vec{r} \times \partial_t \vec{r} ds dt$$

$$= (\hat{y} - \hat{x}) \times (\hat{z} - \hat{x}) ds dt = (\hat{x} + \hat{z} + \hat{y}) ds dt$$

$$\begin{aligned} \int_{\Sigma_4} \vec{E} \cdot d\vec{S} &= \int_D [(x+1) + y - z] ds dt = \int_D [(1-s-t+1) + s - t] ds dt \\ &= 2 \int_0^1 dt \int_0^{1-t} ds (1-t) = 2 \int_0^1 dt (1-t) \int_0^{1-t} ds \\ &= 2 \int_0^1 dt (1-t)(1-t) = 2 \int_0^1 \frac{(1-t)^3}{3} dt = \frac{2}{3} \quad \checkmark \text{Agrees.} \end{aligned}$$

Volume Integrals

(domain = 3d volume)



$$\int \phi \, d\tau$$

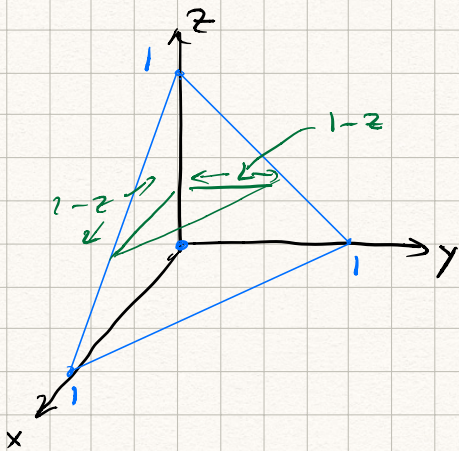
scalar

↑
infinitesimal volume element

Cartesian:

$$d\tau = dx \, dy \, dz = d^3x$$

Ex Total charge in the tetrahedron w/ charge density $\rho = -1$



$$Q = \int_{\text{Tet}} \rho \, d\tau = - \int_{\text{Tet}} d\tau = -(\text{Volume}) = -\frac{1}{6}$$

Calculus derivation:

Area of xy Δ at height z : $\frac{1}{2}(1-z)^2$

$$\int d\tau = \int_0^1 dz \int_{\Delta \text{ at } z} dx \, dy = \int_0^1 dz \, \frac{1}{2}(1-z)^2$$

$$= -\frac{1}{6}(1-z)^3 \Big|_0^1 = \frac{1}{6}$$