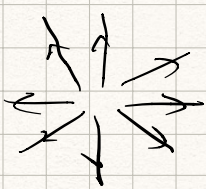


Curl

In Cartesian coordinates:

$$\begin{aligned}\vec{\nabla} \times \vec{v} &= \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \times \left(\hat{x} v_x + \hat{y} v_y + \hat{z} v_z \right) \\&= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \quad \begin{array}{l} \text{must expand such that} \\ \text{derivative acts on components of } \vec{v} \\ \text{"from top row down"} \end{array} \\&= \hat{x} \left(\frac{\partial}{\partial y} v_z - \frac{\partial}{\partial z} v_y \right) + \hat{y} \left(\frac{\partial}{\partial z} v_x - \frac{\partial}{\partial x} v_z \right) + \hat{z} \left(\frac{\partial}{\partial x} v_y - \frac{\partial}{\partial y} v_x \right)\end{aligned}$$

Ex Radial force field $\vec{F} = f(r) \hat{r}$



$$[\vec{\nabla} \times \vec{F}]_x = \frac{\partial}{\partial y} \left(\frac{f(r)}{r} z \right) - \frac{\partial}{\partial z} \left(\frac{f(r)}{r} y \right)$$

$$= \frac{d}{dr} \left(\frac{f(r)}{r} \right) \underbrace{\frac{\partial r}{\partial y}}_{\frac{y}{r}} z + \frac{f(r)}{r} \underbrace{\frac{\partial z}{\partial y}}_0 - \frac{d}{dr} \left(\frac{f(r)}{r} \right) \underbrace{\frac{\partial r}{\partial z}}_{\frac{z}{r}} y - \frac{f(r)}{r} \underbrace{\frac{\partial y}{\partial z}}_0$$

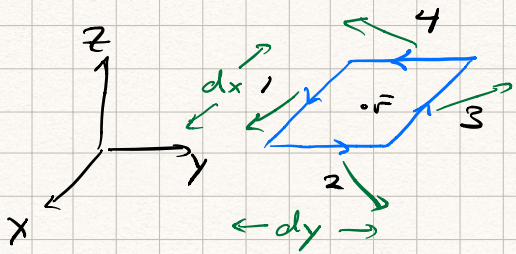
$$= \frac{d}{dr} \left(\frac{f(r)}{r} \right) \left[\frac{y^2}{r} - \frac{z^2}{r} \right] = 0$$

Similar for y, z so

$$\boxed{\vec{\nabla} \times \vec{F} = 0} \quad \text{"irrotational"}$$

Interpretation of Curl

Consider small loop in xy plane:



Think of it as then pipe.
How much flow is going around the loop?

$$\text{Circulation} = \sum_{\text{edges}} \vec{J} \cdot \underbrace{\delta \vec{\ell}}_{\text{flow along edge}}$$

$$= \vec{J}(\vec{r} - \frac{dy}{2} \hat{y}) \cdot dx \hat{x} + \vec{J}(\vec{r} + \frac{dx}{2} \hat{x}) \cdot dy \hat{y} + \vec{J}(\vec{r} + \frac{dx}{2} \hat{x}) \cdot dx (-\hat{x}) + \vec{J}(\vec{r} - \frac{dx}{2} \hat{x}) \cdot dy (-\hat{y})$$

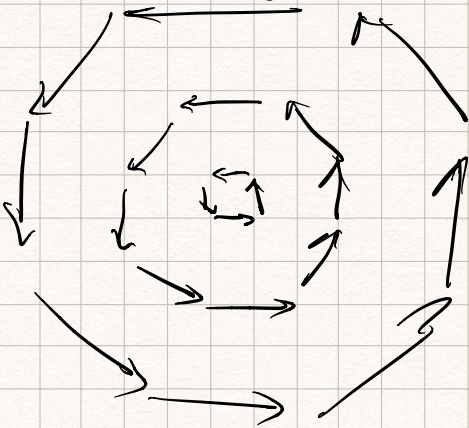
$$= \left[J_x(\vec{r} - \frac{dy}{2} \hat{y}) - J_x(\vec{r} + \frac{dx}{2} \hat{x}) \right] dx + \left[J_y(\vec{r} + \frac{dx}{2} \hat{x}) - J_y(\vec{r} - \frac{dx}{2} \hat{x}) \right] dy$$

$$= -\frac{\partial J_x}{\partial y} dx dy + \frac{\partial J_y}{\partial x} dx dy = (\nabla \times \vec{J}) \cdot \underbrace{(dx dy \hat{z})}_{\substack{\text{oriented area of loop} \\ \text{right handed rule } \begin{matrix} \leftarrow \\ \downarrow \end{matrix}}}$$

$\text{Curl } \vec{J}$ measures circulation (aka vorticity) of flow per unit area.

for velocity field $\vec{v}$, $\nabla \times \vec{v}$ = angular velocity of local flow

Ex Rotating Bucket



Q? Where is curl largest?

It's a constant!

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega \hat{z} \times (x \hat{x} + y \hat{y} + z \hat{z})$$

$$= \omega (x \hat{y} - y \hat{x})$$

$$\nabla \times \vec{v} = \omega [\nabla \times (x \hat{y}) - \nabla \times (y \hat{x})]$$

$$= \omega \hat{z} \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right) = 2\omega \hat{z}$$

Intuition: If you hang a little cork at $\vec{r}$, how much does it spin?

Product Rules

- ∇ is hungry $\Rightarrow$ order matters
- lots of products so lots of rules but all follow from product rules in normal calculus in Cartesian coords

Ex • $\nabla(ab) = \sum_i \hat{e}_i \frac{\partial}{\partial x_i} (ab) = \sum_i \hat{e}_i \left[\frac{\partial a}{\partial x_i} b + a \frac{\partial b}{\partial x_i} \right] = (\nabla a)b + a \nabla b$

• $\nabla \cdot (a\vec{A}) = (\nabla a) \cdot \vec{A} + a \nabla \cdot \vec{A}$
:

Higher Derivatives

Luckily, most physics stops at second orders.

$$\begin{aligned}\nabla f &\rightarrow \boxed{\nabla \cdot (\nabla f) = \nabla^2 f} \\ &\rightarrow \boxed{\nabla \times (\nabla f) = 0}\end{aligned}$$

Laplacian

curl of gradient is zero

$$\nabla \cdot \vec{A} \rightarrow \nabla (\nabla \cdot \vec{A})$$

$$\nabla \times \vec{A} \rightarrow \boxed{\nabla \cdot (\nabla \times \vec{A}) = 0}$$

div of curl is zero

$$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \boxed{\nabla^2 \vec{A}}$$

Vector Laplacian

Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{in Cartesian coordinates}$$

Ex Electrostatics: $\nabla \cdot \vec{E} = \rho/\epsilon_0$

$$\vec{E} = -\nabla \phi$$

$$\Rightarrow \nabla^2 \phi = -\rho/\epsilon_0 \quad \text{Poisson Equation}$$

Vector Laplacian

Defined by: $\nabla^2 \vec{A} = \nabla (\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})$

In Cartesian coords: $\nabla^2 \vec{A}$ acts like scalar Laplacian by component

Ex Maxwell's Equations in Vacuum ($\rho=0, \vec{J}=0$)

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Take curl of $\vec{E}$:

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} (\nabla \times \vec{B}) = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

$$\therefore \nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

Wave Equation for $\vec{E}$!