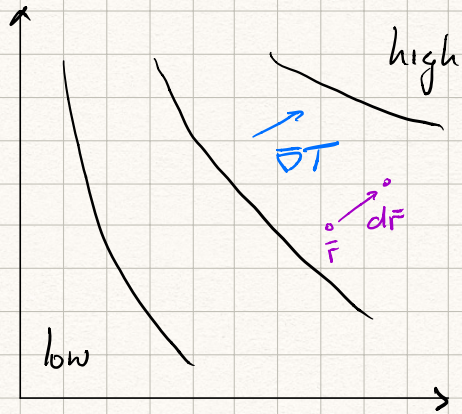


1st Derivative of Scalars: the Gradient



$$T(\bar{r} + d\bar{r}) = T(\bar{r}) + dT$$

$\uparrow$ small displacement $\uparrow$ small change in T

← implicitly depends on $\bar{r}, d\bar{r}$
"total differential"

In Cartesian coordinates:

$$T(x+dx, y+dy, z+dz) = T(x, y, z) + \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

$$\underset{\substack{\uparrow \\ \text{scalar}}}{dT} = \underbrace{\left(\hat{x} \frac{\partial T}{\partial x} + \hat{y} \frac{\partial T}{\partial y} + \hat{z} \frac{\partial T}{\partial z} \right)}_{\text{vector}} \cdot \underset{\substack{\uparrow \\ \text{vector}}}{d\vec{r}}$$

$$\vec{\nabla} T = \text{grad } T \quad \leftarrow \text{vector field}$$

- $\vec{\nabla}T$ points uphill in direction of steepest ascent
- $|\vec{\nabla}T|$ rate of change in that direction
- Must be a vector b/c dT is scalar, $d\vec{r}$ is vector so components of $\vec{\nabla}T$ must transform as vectors.

formally $\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$ is a vector operator

- Operator eats a field and produces another linearly.

∇ is hungry! derivatives act to the right

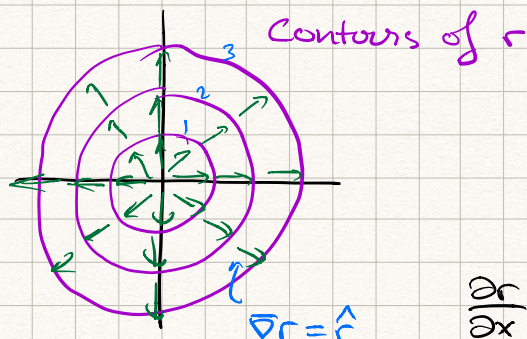
$$f \circ \nabla \neq \nabla f$$

Ex for conservative forces:

$$\bar{F} = -\bar{\nabla} \phi$$

forces $\vec{F}$ point down potential energy ϕ

Ex Gradient of r^n



In Cartesian coords:

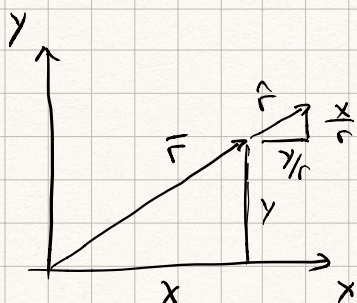
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla r = \hat{x} \frac{\partial r}{\partial x} + \hat{y} \frac{\partial r}{\partial y} + \hat{z} \frac{\partial r}{\partial z}$$

$$\frac{\partial r}{\partial x} = \frac{x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r = \frac{1}{r} (x\hat{x} + y\hat{y} + z\hat{z}) = \frac{\vec{r}}{r} = \hat{r}$$

$$\boxed{\nabla r = \hat{r}}$$



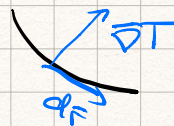
General n : $\frac{\partial r^n}{\partial x} = n r^{n-1} \frac{\partial r}{\partial x}$ (same y, z)

$$\nabla r^n = n r^{n-1} \nabla r = n r^{n-1} \hat{r}$$

Obs • See on contour plot that gradient $\perp$ to contours.

This is always true:

On contour: $0 = dT = (\nabla T) \cdot d\vec{r}$



Ex Coulomb's Law

charge produces potential $\phi \propto 1/r$

$$\vec{F} = -\nabla\left(\frac{1}{r}\right) = -(-1)\frac{1}{r^2} \hat{r} = \frac{\hat{r}}{r^2} \quad \checkmark$$

1st Derivatives of Vectors

Two ways to form product between ∇ and $\vec{v}$:

$$\nabla \cdot \vec{v}$$

scalar

divergence of $\vec{v}$

$$\nabla \times \vec{v}$$

pseudo vector

curl of $\vec{v}$

Divergence

In Cartesian coordinates:

$$\nabla \cdot \vec{v} = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (v_x \hat{x} + v_y \hat{y} + v_z \hat{z})$$

$$= \hat{x} \frac{\partial}{\partial x} (v_x \hat{x}) + \hat{x} \frac{\partial}{\partial x} (v_y \hat{y}) + \dots$$

constant vectors

$$= \cancel{\hat{x} \cdot \hat{x}} \frac{\partial v_x}{\partial x} + \cancel{\hat{x} \cdot \hat{y}} \frac{\partial v_y}{\partial x} + \dots$$

$$\boxed{\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}}$$

Ex Divergence of radius vector

$$\nabla \cdot \vec{r} = \frac{\partial r_x}{\partial x} + \frac{\partial r_y}{\partial y} + \frac{\partial r_z}{\partial z} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

& Divergence of power law central force

$$\vec{F} = r^n \hat{r}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} \left(r^n \frac{x}{r} \right) + \frac{\partial}{\partial y} \left(r^n \frac{y}{r} \right) + \frac{\partial}{\partial z} \left(r^n \frac{z}{r} \right)$$

$$= (n-1) r^{n-2} \frac{\partial r}{\partial x} x + r^{n-1} \frac{\partial x}{\partial x} + \dots$$

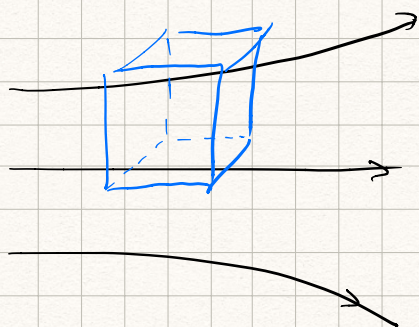
$$= (n-1) r^{n-2} \frac{x^2}{r} + r^{n-1} + \dots$$

$$= (n-1) r^{n-1} + 3 r^{n-1} = (n+2) r^{n-1}$$

Note: Inverse square is special:

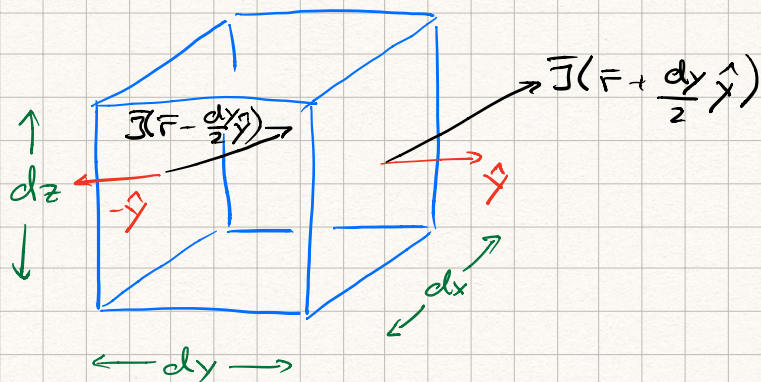
$$\nabla \cdot \left(\frac{\hat{r}}{r^2} \right) = \begin{cases} 0 & r > 0 \\ \infty & r = 0 \end{cases} \quad \text{ill defined (will return to this)}$$

Interpretation of Divergence



Think of $\vec{J}$ as flow of some fluid (eg. $\vec{J} = \rho \vec{v}$)

Consider a small box around $\vec{r}$:



Net flux of fluid out of box:

$$\sum_{\text{sides}} \text{flow out of side} = \underbrace{\vec{J}(\vec{r} + \frac{dy}{2}\hat{y}) \cdot (\underbrace{dx\,dz}_{\text{area of } +y \text{ side}} \underbrace{\hat{y}}_{\text{unit outward normal}})}_{\text{flux per unit area at } +y \text{ side}} + \vec{J}(\vec{r} - \frac{dy}{2}\hat{y}) \cdot (dx\,dz\,(-\hat{y}))$$

+ ... (x, z sides)

$$= \left(J_y(\vec{r} + \frac{dy}{2}\hat{y}) - J_y(\vec{r} - \frac{dy}{2}\hat{y}) \right) dx\,dz + \dots$$

$$= \left(\cancel{J_y(\vec{r})} + \frac{\partial J_y}{\partial y} \frac{dy}{2} + \dots - \left[\cancel{J_y(\vec{r})} + \frac{\partial J_y}{\partial y} \left(-\frac{dy}{2}\right) + \dots \right] \right) dx\,dz + \dots$$

$$= \frac{\partial J_y}{\partial y} dx\,dy\,dz + \text{x, z terms}$$

$$= (\nabla \cdot \vec{J}) \underbrace{dx\,dy\,dz}_{\text{volume of box}}$$

Divergence measures net out flow from small box, per unit volume.

Rem If $\vec{J}$ represents flow of conserved quantity, such as mass of water or charge, then density ρ satisfies the continuity equation:

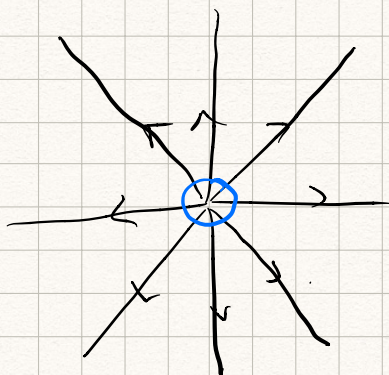
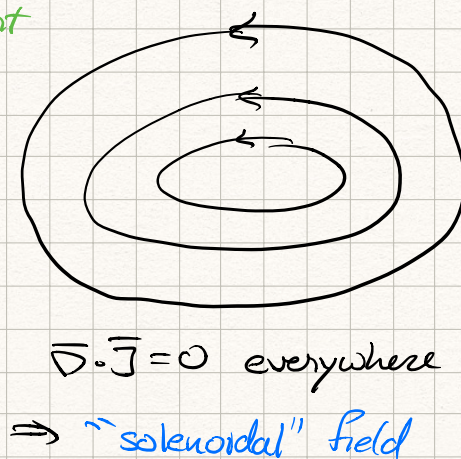
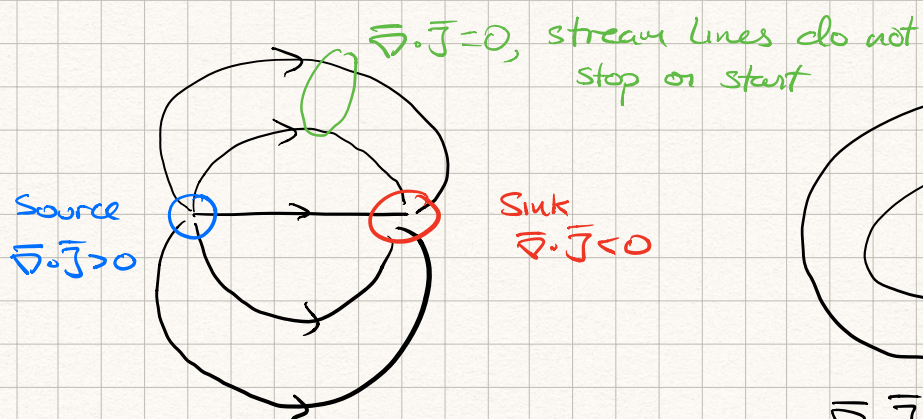
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$$

amount of stuff in small box changes in time by the net flow into the box

Q? Where is divergence large, zero, negative?

↑ ↑ ↑ ↑
↑ ↑ ↑ ↑
 $\vec{J} = \text{const}$
 $\nabla \cdot \vec{J} = 0$

↑ ↑ ↑
↑ ↑ ↑
 $\vec{J} = x\hat{y}$
 $\nabla \cdot \vec{J} = 0$



$\nabla \cdot \vec{J} = 0$ except @ origin