

## Problem Set 8: Fundamental Theorems, Potentials, and Curvilinear Coordinates

### 1. Integration by parts for curl.

Start from the product rule for the divergence of a cross product:

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

*Proof of this identity:*

$$\begin{aligned} \vec{\nabla} \cdot (\vec{A} \times \vec{B}) &= \epsilon_{ijk} \partial_i (A_j B_k) = \epsilon_{ijk} (\partial_i A_j) B_k + \epsilon_{ijk} A_j (\partial_i B_k) \\ &= \epsilon_{kij} (\partial_i A_j) B_k - \epsilon_{jik} A_j (\partial_i B_k) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \end{aligned}$$

where we used  $\epsilon_{ijk} = \epsilon_{kij} = -\epsilon_{jik}$ .

Integrate both sides over a volume  $\mathcal{V}$ :

$$\int_{\mathcal{V}} \vec{\nabla} \cdot (\vec{A} \times \vec{B}) dv = \int_{\mathcal{V}} \vec{B} \cdot (\vec{\nabla} \times \vec{A}) dv - \int_{\mathcal{V}} \vec{A} \cdot (\vec{\nabla} \times \vec{B}) dv$$

Apply the divergence theorem to the left-hand side:

$$\oint_{\mathcal{S}} (\vec{A} \times \vec{B}) \cdot d\vec{\sigma} = \int_{\mathcal{V}} \vec{B} \cdot (\vec{\nabla} \times \vec{A}) dv - \int_{\mathcal{V}} \vec{A} \cdot (\vec{\nabla} \times \vec{B}) dv$$

Rearranging:

$$\boxed{\int_{\mathcal{V}} \vec{B} \cdot (\vec{\nabla} \times \vec{A}) dv = \int_{\mathcal{V}} \vec{A} \cdot (\vec{\nabla} \times \vec{B}) dv + \oint_{\mathcal{S}} (\vec{A} \times \vec{B}) \cdot d\vec{\sigma}} \quad \square$$

The derivative (curl) is transferred from  $\vec{A}$  to  $\vec{B}$  at the cost of a boundary term (surface integral).

### 2. Radial force field.

$$\vec{F} = r^n \hat{r} = r^{n-1} (x\hat{x} + y\hat{y} + z\hat{z}) = (x^2 + y^2 + z^2)^{\frac{n-1}{2}} (x\hat{x} + y\hat{y} + z\hat{z})$$

(a) Curl:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ x r^{n-1} & y r^{n-1} & z r^{n-1} \end{vmatrix}$$

The  $\hat{x}$  component is:

$$\frac{\partial}{\partial y} (z r^{n-1}) - \frac{\partial}{\partial z} (y r^{n-1}) = z(n-1)r^{n-3}y - y(n-1)r^{n-3}z = 0$$

By symmetry, all components vanish:  $\boxed{\vec{\nabla} \times \vec{F} = 0}$

(b) Scalar potential:  $\vec{F} = -\vec{\nabla}\phi$

$$\phi(x, y, z) = - \int_0^{\vec{r}} \vec{F} \cdot d\vec{r}' = - \int_0^r r'^n dr' = - \frac{r'^{n+1}}{n+1} \Big|_0^r = \boxed{-\frac{(x^2 + y^2 + z^2)^{\frac{n+1}{2}}}{n+1}}$$

(for  $n \neq -1$ ; the case  $n = -1$  gives  $\phi = -\ln r$ )

(c) For  $\phi$  to diverge at both origin and infinity:

- At origin ( $r \rightarrow 0$ ): need  $\frac{n+1}{2} < 0$ , i.e.,  $n < -1$
- At infinity ( $r \rightarrow \infty$ ): need  $\frac{n+1}{2} > 0$ , i.e.,  $n > -1$

These cannot both be satisfied, except for the special case  $\boxed{n = -1}$ , where  $\phi = -\ln r$  diverges logarithmically at both limits.

### 3. Parallel plate capacitor.

(a) Check Laplace equation for  $\phi = Vz/d$ :

$$\nabla^2 \phi = \vec{\nabla} \cdot (\vec{\nabla} \phi) = \vec{\nabla} \cdot \left( \frac{V}{d} \hat{z} \right) = \frac{\partial}{\partial z} \left( \frac{V}{d} \right) = 0 \quad \checkmark$$

Check boundary conditions:

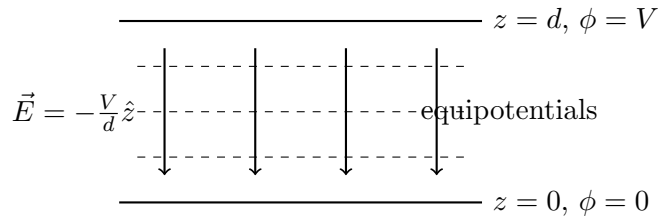
$$\begin{aligned} \phi(z=0) &= \frac{V \cdot 0}{d} = 0 \quad \checkmark \\ \phi(z=d) &= \frac{V \cdot d}{d} = V \quad \checkmark \end{aligned}$$

(b) Electric field:

$$\vec{E} = -\vec{\nabla}\phi = -\frac{\partial}{\partial z} \left( \frac{Vz}{d} \right) \hat{z} = \boxed{-\frac{V}{d} \hat{z}}$$

The field is uniform, pointing from high to low potential (from  $z = d$  to  $z = 0$ ).

(c) Sketch:



The electric field lines are vertical (pointing downward), and the equipotentials are horizontal planes at constant  $z$ .

### 4. Vector potential for a uniform magnetic field.

$\vec{A} = \frac{1}{2} B_0 \rho \hat{\phi}$  in cylindrical coordinates.

(a) With  $A_\rho = 0$ ,  $A_\phi = \frac{1}{2} B_0 \rho$ ,  $A_z = 0$ , the cylindrical curl gives only a  $\hat{z}$  component:

$$\vec{B} = \frac{1}{\rho} \frac{\partial(\rho A_\phi)}{\partial \rho} \hat{z} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{1}{2} B_0 \rho^2 \right) \hat{z} = \boxed{B_0 \hat{z}}$$

This is a uniform magnetic field in the  $z$ -direction.

(b) On a disk in the  $xy$ -plane,  $d\vec{\sigma} = \rho d\rho d\phi \hat{e}_z$ :

$$\Phi = \int_0^{2\pi} d\phi \int_0^R d\rho \rho B_0 = B_0 \cdot 2\pi \cdot \frac{R^2}{2} = \boxed{B_0 \pi R^2}$$

(c) By Stokes' theorem,  $\Phi = \oint_C \vec{A} \cdot d\vec{l}$  where  $C$  is the boundary circle  $\rho = R$ .

On the circle:  $d\vec{l} = R d\phi \hat{e}_\phi$  and  $\vec{A} = \frac{1}{2} B_0 R \hat{e}_\phi$ .

$$\Phi = \oint_C \vec{A} \cdot d\vec{l} = \int_0^{2\pi} \frac{1}{2} B_0 R \cdot R d\phi = \boxed{B_0 \pi R^2} \quad \checkmark$$

This agrees with the direct calculation in part (b), confirming Stokes' theorem.

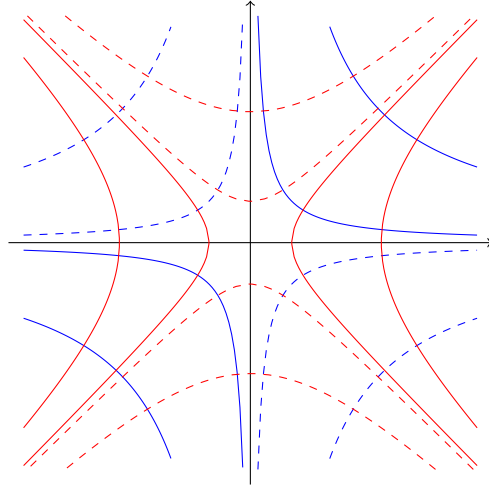
## 5. Curvilinear coordinate system.

$$xy = u, \quad x^2 - y^2 = v, \quad z = z$$

(a) The three families of coordinate surfaces:

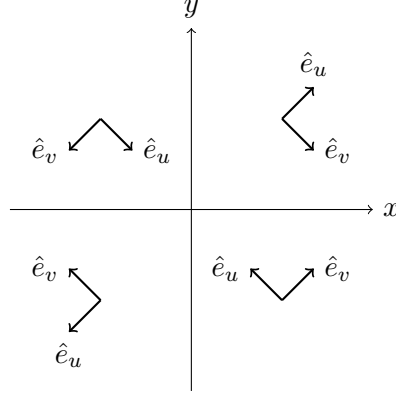
- $z = z_0$ : horizontal planes parallel to the  $xy$ -plane
- $u = u_0 \Rightarrow xy = u_0$ : hyperbolic cylinders (rectangular hyperbolas in  $xy$ , extending in  $z$ ) with asymptotes along the  $x$ - and  $y$ -axes
- $v = v_0 \Rightarrow x^2 - y^2 = v_0$ : hyperbolic cylinders with asymptotes along  $y = \pm x$

(b) Sketch of coordinate surfaces in the  $xy$  plane:



Blue curves:  $u = \text{const}$  (rectangular hyperbolas  $xy = u$  with asymptotes along the axes); solid for  $u > 0$  (quadrants I, III), dashed for  $u < 0$  (quadrants II, IV). Red curves:  $v = \text{const}$  (hyperbolas  $x^2 - y^2 = v$ ); solid for  $v > 0$  (opening along  $x$ -axis), dashed for  $v < 0$  (opening along  $y$ -axis). The two families are everywhere orthogonal. Shown:  $u, v = \pm 0.3, \pm 3$ .

(c) The local basis vectors  $\hat{e}_u$  and  $\hat{e}_v$  point in the directions of  $\vec{\nabla}u$  and  $\vec{\nabla}v$  respectively (perpendicular to the corresponding constant-coordinate curves, in the direction of increase). Since  $\vec{\nabla}u = y\hat{x} + x\hat{y}$  and  $\vec{\nabla}v = 2x\hat{x} - 2y\hat{y}$ :



- (d)  $\hat{e}_u \times \hat{e}_v \propto (y, x, 0) \times (x, -y, 0) = (0, 0, -y^2 - x^2) \propto -\hat{z}$ , so  $\hat{e}_u \times \hat{e}_v = -\hat{e}_z$  and the coordinate system is **left-handed**.

## 6. Derivatives of cylindrical unit vectors.

Write the local basis in terms of Cartesian basis:

$$\begin{aligned}\hat{e}_\rho &= \cos \phi \hat{x} + \sin \phi \hat{y} \\ \hat{e}_\phi &= -\sin \phi \hat{x} + \cos \phi \hat{y} \\ \hat{e}_z &= \hat{z}\end{aligned}$$

Taking derivatives:

$$\begin{aligned}\frac{\partial \hat{e}_\rho}{\partial \phi} &= -\sin \phi \hat{x} + \cos \phi \hat{y} = \boxed{\hat{e}_\phi} \\ \frac{\partial \hat{e}_\phi}{\partial \phi} &= -\cos \phi \hat{x} - \sin \phi \hat{y} = \boxed{-\hat{e}_\rho}\end{aligned}$$

All other derivatives vanish:

$$\frac{\partial \hat{e}_\rho}{\partial \rho} = \frac{\partial \hat{e}_\rho}{\partial z} = \frac{\partial \hat{e}_\phi}{\partial \rho} = \frac{\partial \hat{e}_\phi}{\partial z} = \frac{\partial \hat{e}_z}{\partial \rho} = \frac{\partial \hat{e}_z}{\partial \phi} = \frac{\partial \hat{e}_z}{\partial z} = 0$$

## 7. Rotating rigid body in spherical coordinates.

$\vec{\omega} = \omega \hat{z}$  and  $\vec{r} = r \hat{e}_r$  in spherical coordinates.

- (a)  $\vec{v} = \vec{\omega} \times \vec{r}$

In spherical coordinates,  $\hat{z} = \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta$ .

$$\begin{aligned}\vec{v} &= \omega \hat{z} \times r \hat{e}_r = \omega r (\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta) \times \hat{e}_r \\ &= \omega r (-\sin \theta) (\hat{e}_\theta \times \hat{e}_r) = \omega r \sin \theta \hat{e}_\phi = \boxed{\omega r \sin \theta \hat{e}_\phi}\end{aligned}$$

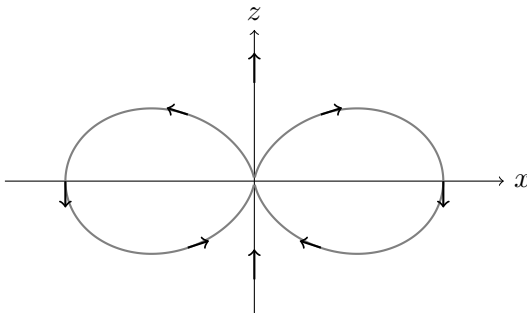
- (b) Using the spherical curl formula with  $v_r = v_\theta = 0$  and  $v_\phi = \omega r \sin \theta$ :

$$\vec{\nabla} \times \vec{v} = \frac{1}{r \sin \theta} \frac{\partial (v_\phi \sin \theta)}{\partial \theta} \hat{e}_r - \frac{1}{r} \frac{\partial (r v_\phi)}{\partial r} \hat{e}_\theta = \boxed{2\omega \cos \theta \hat{e}_r - 2\omega \sin \theta \hat{e}_\theta = 2\omega \hat{z}}$$

8. Force field in spherical coordinates.

$$\vec{F} = \hat{e}_r \frac{2P \cos \theta}{r^3} + \hat{e}_\theta \frac{P \sin \theta}{r^3}$$

(a) Sketch (like the field of a magnetic dipole):



(b) Using the spherical curl formula:

$$\vec{\nabla} \times \vec{F} = \frac{1}{r \sin \theta} \begin{vmatrix} \hat{e}_r & r \hat{e}_\theta & r \sin \theta \hat{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ \frac{2P \cos \theta}{r^3} & \frac{P \sin \theta}{r^2} & 0 \end{vmatrix} = \boxed{0}$$

Yes,  $\vec{F}$  is **conservative**.

(c) On  $\theta = \pi/2$  plane,  $r = 1$ :  $\vec{F} \cdot \hat{\phi} = 0$  (since  $F_\phi = 0$ ), and  $d\vec{r} = r \sin \theta d\phi \hat{e}_\phi = d\phi \hat{e}_\phi$ .

$$\oint \vec{F} \cdot d\vec{r} = \boxed{0}$$

Consistent with  $\vec{F}$  being conservative.

(d) Since  $\vec{F} = -\vec{\nabla} \phi$ , we need  $F_r = -\partial_r \phi$  and  $F_\theta = -\frac{1}{r} \partial_\theta \phi$ .

Integrating  $F_r = -\partial_r \phi$ :

$$\phi = - \int \frac{2P \cos \theta}{r^3} dr = \frac{P \cos \theta}{r^2}$$

Check:  $F_\theta = -\frac{1}{r} \partial_\theta \phi = -\frac{1}{r} \left( \frac{-P \sin \theta}{r^2} \right) = \frac{P \sin \theta}{r^3} \checkmark$

$$\boxed{\phi = \frac{P \cos \theta}{r^2}}$$