

Problem Set 6: Vector Derivatives and Vector Calculus

1. Gradients of scalar fields.

(a) $f(\vec{r}) = r^{-3/2}$ where $r = \sqrt{x^2 + y^2 + z^2}$.

Using the chain rule and $\vec{\nabla}r = \hat{r}$:

$$\vec{\nabla}f = \frac{df}{dr}\vec{\nabla}r = -\frac{3}{2}r^{-5/2}\hat{r} = \boxed{-\frac{3}{2}r^{-5/2}\hat{r}}$$

(b) $g(\vec{r}) = x^2 + y^3 + z^4$

$$\vec{\nabla}g = \frac{\partial g}{\partial x}\hat{x} + \frac{\partial g}{\partial y}\hat{y} + \frac{\partial g}{\partial z}\hat{z} = \boxed{2x\hat{x} + 3y^2\hat{y} + 4z^3\hat{z}}$$

(c) At the point (2, 2, 1):

$$r = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

$$|\vec{\nabla}f| = \frac{3}{2}r^{-5/2} = \frac{3}{2} \cdot 3^{-5/2} = \boxed{\frac{1}{6\sqrt{3}}}$$

2. Gradient of r_{12} .

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 \text{ and } r_{12} = |\vec{r}_{12}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

$$\begin{aligned}\vec{\nabla}_1 r_{12} &= \left(\hat{x} \frac{\partial}{\partial x_1} + \hat{y} \frac{\partial}{\partial y_1} + \hat{z} \frac{\partial}{\partial z_1} \right) \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \\ &= \frac{1}{r_{12}} ((x_1 - x_2)\hat{x} + (y_1 - y_2)\hat{y} + (z_1 - z_2)\hat{z}) \\ &= \frac{\vec{r}_{12}}{r_{12}} = \hat{r}_{12}\end{aligned}$$

This is the unit vector $\hat{r}_{12}$ pointing in the direction of $\vec{r}_{12}$. $\square$

3. Product rule for gradient.

We want to show $\vec{\nabla}(uv) = u\vec{\nabla}v + v\vec{\nabla}u$.

The $\hat{x}_i$ component of the left-hand side is:

$$\hat{x}_i : \quad \frac{\partial}{\partial x_i}(uv) = u \frac{\partial v}{\partial x_i} + v \frac{\partial u}{\partial x_i}$$

This equals the $\hat{x}_i$ component of $(u\vec{\nabla}v + v\vec{\nabla}u)$, so:

$$\boxed{\vec{\nabla}(uv) = u\vec{\nabla}v + v\vec{\nabla}u} \quad \square$$

4. Linearity of gradient.

For constants a, b and scalar fields F, G :

$$\begin{aligned}\vec{\nabla}(aF + bG) &= \hat{x}_i \frac{\partial}{\partial x_i} (aF + bG) \\ &= \hat{x}_i \left(a \frac{\partial F}{\partial x_i} + b \frac{\partial G}{\partial x_i} \right) \\ &= a \hat{x}_i \frac{\partial F}{\partial x_i} + b \hat{x}_i \frac{\partial G}{\partial x_i} \\ &= a \vec{\nabla} F + b \vec{\nabla} G \quad \square\end{aligned}$$

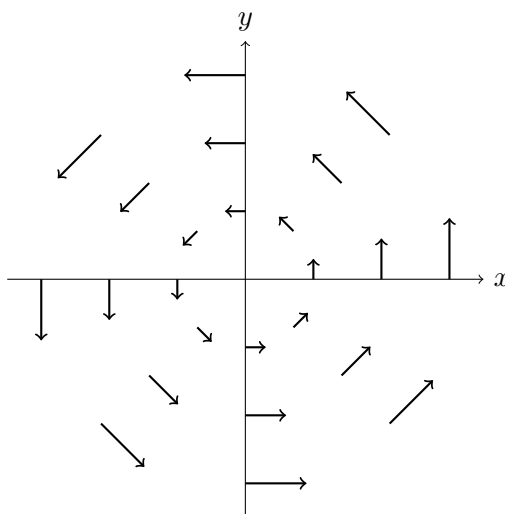
5. Rotating rigid body.

$\vec{\omega} = \omega \hat{z}$ and $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

(a)

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \hat{x}(-y\omega) + \hat{y}(x\omega) + \hat{z}(0) = \boxed{\omega(x\hat{y} - y\hat{x})}$$

(b) Arrow plot (setting $\omega = 1$):



The vectors circulate counterclockwise around the origin, with magnitude $|\vec{v}| = \omega\sqrt{x^2 + y^2}$ increasing with distance from the axis.

(c) Divergence:

$$\vec{\nabla} \cdot \vec{v} = \omega \left[\frac{\partial(-y)}{\partial x} + \frac{\partial(x)}{\partial y} \right] = \omega[0 + 0] = \boxed{0}$$

The velocity field is solenoidal (incompressible flow).

(d) Curl:

$$\vec{\nabla} \times \vec{v} = \omega \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \omega \hat{z} \left(\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right) = \boxed{2\omega \hat{z}}$$

6. **Divergence and curl of $\vec{v} = r\hat{x}$.**

$$\vec{v} = r\hat{x} = \sqrt{x^2 + y^2 + z^2} \hat{x}$$

Divergence:

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \boxed{\frac{x}{r}}$$

Curl:

$$\begin{aligned} \vec{\nabla} \times \vec{v} &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ r & 0 & 0 \end{vmatrix} = \hat{x}(0) - \hat{y}\left(-\frac{\partial r}{\partial z}\right) + \hat{z}\left(-\frac{\partial r}{\partial y}\right) \\ &= \frac{z}{r}\hat{y} - \frac{y}{r}\hat{z} = \boxed{\frac{z}{r}\hat{y} - \frac{y}{r}\hat{z}} \end{aligned}$$

7. **Curl of gradient and divergence of curl.**

(a) $\vec{\nabla}\phi = (\partial_x\phi)\hat{x} + (\partial_y\phi)\hat{y} + (\partial_z\phi)\hat{z}$

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla}\phi) &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \partial_x\phi & \partial_y\phi & \partial_z\phi \end{vmatrix} \\ &= \hat{x}(\partial_y\partial_z\phi - \partial_z\partial_y\phi) + \hat{y}(\partial_z\partial_x\phi - \partial_x\partial_z\phi) + \hat{z}(\partial_x\partial_y\phi - \partial_y\partial_x\phi) \\ &= \boxed{0} \end{aligned}$$

due to interchangeability of mixed partials (e.g., $\partial_x\partial_y = \partial_y\partial_x$).

Index notation (using Einstein summation convention):

$$[\vec{\nabla} \times (\vec{\nabla}\phi)]_i = \epsilon_{ijk} \partial_j \partial_k \phi = 0$$

because ϵ_{ijk} is antisymmetric in j, k while $\partial_j \partial_k \phi$ is symmetric in j, k , and the contraction of a symmetric and antisymmetric tensor vanishes.

(b) $\vec{\nabla} \times \vec{A} = \hat{x}(\partial_y A_z - \partial_z A_y) + \hat{y}(\partial_z A_x - \partial_x A_z) + \hat{z}(\partial_x A_y - \partial_y A_x)$

$$\begin{aligned} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) &= \partial_x(\partial_y A_z - \partial_z A_y) + \partial_y(\partial_z A_x - \partial_x A_z) + \partial_z(\partial_x A_y - \partial_y A_x) \\ &= \partial_x \partial_y A_z - \partial_x \partial_z A_y + \partial_y \partial_z A_x - \partial_y \partial_x A_z + \partial_z \partial_x A_y - \partial_z \partial_y A_x \\ &= \boxed{0} \end{aligned}$$

Each term cancels with another due to mixed partial interchangeability.

Index notation (using Einstein summation convention):

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \partial_i (\epsilon_{ijk} \partial_j A_k) = \epsilon_{ijk} \partial_i \partial_j A_k = 0$$

because ϵ_{ijk} is antisymmetric in i, j while $\partial_i \partial_j$ is symmetric in i, j .

8. **Gradient of harmonic function.**

Given $\nabla^2 \phi = 0$, i.e., $\vec{\nabla} \cdot (\vec{\nabla} \phi) = 0$.

Divergence: $\vec{\nabla} \cdot (\vec{\nabla} \phi) = \nabla^2 \phi = 0$ ✓

Curl: $\vec{\nabla} \times (\vec{\nabla} \phi) = 0$ from Problem 7(a). ✓

Therefore $\vec{\nabla} \phi$ has **neither curl nor divergence**.

9. **Cross product of irrotational fields is solenoidal.**

Given: $\vec{\nabla} \times \vec{u} = 0$ and $\vec{\nabla} \times \vec{v} = 0$.

In index notation (Einstein summation convention), $(\vec{u} \times \vec{v})_i = \epsilon_{ijk} u_j v_k$, so:

$$\begin{aligned} \vec{\nabla} \cdot (\vec{u} \times \vec{v}) &= \partial_i (\epsilon_{ijk} u_j v_k) = \epsilon_{ijk} (\partial_i u_j) v_k + \epsilon_{ijk} u_j (\partial_i v_k) \\ &= (\vec{\nabla} \times \vec{u})_k v_k + u_j (-1)(\vec{\nabla} \times \vec{v})_j = \boxed{0} \end{aligned}$$

where we used $\epsilon_{ijk} \partial_i u_j = (\vec{\nabla} \times \vec{u})_k$ and $\epsilon_{ijk} \partial_i v_k = -\epsilon_{ikj} \partial_i v_k = -(\vec{\nabla} \times \vec{v})_j$.

10. **Fields with equal curls.**

Given: $\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{B}$.

(a) Let $\vec{A} = \vec{B} + \vec{a}$ where $\vec{a}$ is a constant vector.

Then $\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{B} + \vec{\nabla} \times \vec{a} = \vec{\nabla} \times \vec{B} + 0 = \vec{\nabla} \times \vec{B}$. ✓

(b) Let $\vec{A} = \vec{B} + \vec{\nabla} \phi$ for some scalar field ϕ .

Then $\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{B} + \vec{\nabla} \times (\vec{\nabla} \phi) = \vec{\nabla} \times \vec{B} + 0 = \vec{\nabla} \times \vec{B}$. ✓

(using Problem 7(a))