

One More Complex Contour Integration

1. Contour integration of $e^{iz}/(z^2 + 1)$.

- (a) The denominator factors as $z^2 + 1 = (z - i)(z + i)$, giving simple poles at $z = \pm i$. Only $z_0 = i$ lies in the upper half-plane (and hence inside C for $R > 1$).

The residue at this simple pole:

$$\text{Res}_{z=i} f(z) = \lim_{z \rightarrow i} (z - i) \frac{e^{iz}}{(z - i)(z + i)} = \frac{e^{i \cdot i}}{i + i} = \frac{e^{-1}}{2i}.$$

By the residue theorem,

$$\oint_C f(z) dz = 2\pi i \times \frac{e^{-1}}{2i} = \frac{\pi}{e}.$$

- (b) On Γ_R , write $z = Re^{i\theta}$ with $0 \leq \theta \leq \pi$. Then

$$|e^{iz}| = |e^{iR(\cos \theta + i \sin \theta)}| = e^{-R \sin \theta}.$$

Since $\sin \theta \geq 0$ for $\theta \in [0, \pi]$, we have $|e^{iz}| \leq 1$ throughout the upper semicircle. Therefore

$$\left| \int_{\Gamma_R} f(z) dz \right| \leq \int_0^\pi \frac{|e^{iz}|}{|z^2 + 1|} R d\theta \leq \int_0^\pi \frac{R d\theta}{R^2 - 1} = \frac{\pi R}{R^2 - 1} \xrightarrow{R \rightarrow \infty} 0.$$

- (c) On the real axis, $f(x) = e^{ix}/(x^2 + 1)$. The contour splits as

$$\oint_C f(z) dz = \int_{-R}^R \frac{e^{ix}}{x^2 + 1} dx + \int_{\Gamma_R} f(z) dz.$$

Taking $R \rightarrow \infty$ and using (a) and (b):

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 1} dx = \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx + i \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 1} dx,$$

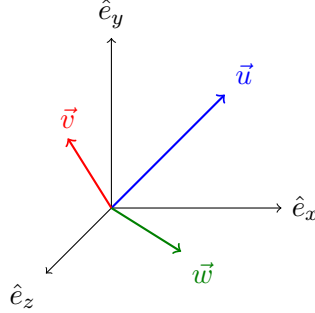
where we used Euler's formula $e^{ix} = \cos x + i \sin x$. Since π/e is real, the imaginary part vanishes (as expected: $\sin x/(x^2 + 1)$ is odd), and

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx = \frac{\pi}{e}.$$

Vectors

2. Three vectors.

(a) Sketch with all vectors originating at the origin:



(b) Length of each vector:

$$|\vec{u}| = \sqrt{u_x^2 + u_y^2 + u_z^2} = \sqrt{1 + 1 + 0} = \boxed{\sqrt{2}}$$

$$|\vec{v}| = \sqrt{0 + 1 + 1} = \boxed{\sqrt{2}}$$

$$|\vec{w}| = \sqrt{1 + 0 + 1} = \boxed{\sqrt{2}}$$

(c) The angles between $\{\vec{u}, \vec{v}\}$, $\{\vec{u}, \vec{w}\}$, and $\{\vec{v}, \vec{w}\}$ are all equal. Computing one:

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \theta$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{(\hat{e}_x + \hat{e}_y) \cdot (\hat{e}_y + \hat{e}_z)}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}$$

Therefore:

$$\boxed{\theta = \frac{\pi}{3}}$$

(d) Volume of parallelepiped:

$$\begin{aligned} \text{Vol} &= |\vec{u} \cdot (\vec{v} \times \vec{w})| \\ &= |(\hat{e}_x + \hat{e}_y) \cdot ((\hat{e}_y + \hat{e}_z) \times (\hat{e}_x + \hat{e}_z))| \\ &= |(\hat{e}_x + \hat{e}_y) \cdot (\hat{e}_y \times \hat{e}_x + \hat{e}_y \times \hat{e}_z + \hat{e}_z \times \hat{e}_x + \hat{e}_z \times \hat{e}_z)| \\ &= |(\hat{e}_x + \hat{e}_y) \cdot (-\hat{e}_z + \hat{e}_x + \hat{e}_y + 0)| \\ &= |1 + 1| = \boxed{2} \end{aligned}$$

3. Vector making equal angles with coordinate axes.

If $\vec{A}$ makes equal angles with the coordinate axes, then $\vec{A} \cdot \hat{x} = \vec{A} \cdot \hat{y} = \vec{A} \cdot \hat{z}$, which implies $A_x = A_y = A_z$.

The magnitude is:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{3A_x^2} = \sqrt{3}|A_x| = 1.732 \approx \sqrt{3}$$

Therefore $|A_x| = 1$, so:

$$\vec{A} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

4. Orthonormal basis.

$$\vec{e}'_1 = \frac{1}{9} \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix} \quad \vec{e}'_2 = \frac{1}{9} \begin{pmatrix} -7 \\ 4 \\ 4 \end{pmatrix} \quad \vec{e}'_3 = \frac{1}{9} \begin{pmatrix} -4 \\ -8 \\ 1 \end{pmatrix}$$

(a) The basis is orthonormal if $\vec{e}'_i \cdot \vec{e}'_j = \delta_{ij}$. By explicit calculation:

Normalization:

$$\begin{aligned} \vec{e}'_1 \cdot \vec{e}'_1 &= \frac{1}{81}(16 + 1 + 64) = \frac{81}{81} = 1 \\ \vec{e}'_2 \cdot \vec{e}'_2 &= \frac{1}{81}(49 + 16 + 16) = \frac{81}{81} = 1 \\ \vec{e}'_3 \cdot \vec{e}'_3 &= \frac{1}{81}(16 + 64 + 1) = \frac{81}{81} = 1 \end{aligned}$$

Orthogonality:

$$\begin{aligned} \vec{e}'_1 \cdot \vec{e}'_2 &= \frac{1}{81}(4 \cdot (-7) + (-1) \cdot 4 + 8 \cdot 4) = \frac{1}{81}(-28 - 4 + 32) = 0 \\ \vec{e}'_1 \cdot \vec{e}'_3 &= \frac{1}{81}(4 \cdot (-4) + (-1) \cdot (-8) + 8 \cdot 1) = \frac{1}{81}(-16 + 8 + 8) = 0 \\ \vec{e}'_2 \cdot \vec{e}'_3 &= \frac{1}{81}((-7) \cdot (-4) + 4 \cdot (-8) + 4 \cdot 1) = \frac{1}{81}(28 - 32 + 4) = 0 \end{aligned}$$

Therefore $\{\vec{e}'_i\}$ forms an **orthonormal basis**.

(b) $\vec{w} = (1, 2, 3)^T$ in standard basis. The components on the $\vec{e}'_i$ basis are:

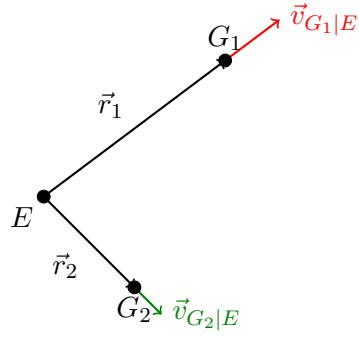
$$\begin{aligned} w'^1 &= \vec{w} \cdot \vec{e}'_1 = \frac{1}{9}(4 \cdot 1 + (-1) \cdot 2 + 8 \cdot 3) = \frac{1}{9}(4 - 2 + 24) = \boxed{\frac{26}{9}} \\ w'^2 &= \vec{w} \cdot \vec{e}'_2 = \frac{1}{9}((-7) \cdot 1 + 4 \cdot 2 + 4 \cdot 3) = \frac{1}{9}(-7 + 8 + 12) = \boxed{\frac{13}{9}} \\ w'^3 &= \vec{w} \cdot \vec{e}'_3 = \frac{1}{9}((-4) \cdot 1 + (-8) \cdot 2 + 1 \cdot 3) = \frac{1}{9}(-4 - 16 + 3) = \boxed{-\frac{17}{9}} \end{aligned}$$

5. Hubble's law.

Use the notation $\vec{v}_{A|B}$ to denote the velocity vector of point A as seen from point B .

Hubble's law says that galaxies G_1 and G_2 have velocities with respect to Earth:

$$\vec{v}_{G_1|E} = H_0 \vec{r}_1, \quad \vec{v}_{G_2|E} = H_0 \vec{r}_2$$



Let's sit on galaxy G_1 : the velocity of galaxy G_2 with respect to G_1 is:

$$\begin{aligned}\vec{v}_{G_2|G_1} &= \vec{v}_{G_2|E} - \vec{v}_{G_1|E} \\ &= H_0 \vec{r}_2 - H_0 \vec{r}_1 \\ &= H_0 (\vec{r}_2 - \vec{r}_1)\end{aligned}$$

But $\vec{r}_2 - \vec{r}_1$ is the position of G_2 as seen from G_1 , so Hubble's law is respected from the point of view of an observer on G_1 .

This shows that **every galaxy sees all other galaxies receding according to Hubble's law**, so we are not at a special location.