

Problem Set 3: Complex Numbers

Note: Don't forget the phase difference of π in cases where the real part is negative.

1. Cartesian and polar forms.

(a)

$$z_1 = \frac{2}{1+i} = \frac{2(1-i)}{(1+i)(1-i)} = \frac{2(1-i)}{2} = 1-i$$

Modulus: $|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Phase: $\theta = \arctan(-1) = -\pi/4$.

$$z_1 = 1-i = \sqrt{2} e^{-i\pi/4}$$

(b)

$$z_2 = (1 + \sqrt{3}i)^2 = 1 + 2\sqrt{3}i + 3i^2 = 1 + 2\sqrt{3}i - 3 = -2 + 2\sqrt{3}i$$

Modulus: $|z_2| = |1 + \sqrt{3}i|^2 = 4$. Phase: $\theta = \arctan(-\sqrt{3}) + \pi = -\pi/3 + \pi = 2\pi/3$.

$$z_2 = -2 + 2\sqrt{3}i = 4 e^{i2\pi/3}$$

(c)

$$z_3 = \frac{1 + \sqrt{3}i}{1 - \sqrt{3}i} = \frac{(1 + \sqrt{3}i)^2}{(1 - \sqrt{3}i)(1 + \sqrt{3}i)} = \frac{1 + 2\sqrt{3}i - 3}{4} = \frac{-2 + 2\sqrt{3}i}{4} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

Modulus: $|z_3| = \frac{|1+\sqrt{3}i|}{|1-\sqrt{3}i|} = 1$. Phase: $\theta = \arctan(-\sqrt{3}) + \pi = 2\pi/3$.

$$z_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{i2\pi/3}$$

(d)

$$z_4 = \frac{1-i}{1-\sqrt{3}i} = \frac{(1-i)(1+\sqrt{3}i)}{(1-\sqrt{3}i)(1+\sqrt{3}i)} = \frac{1 + \sqrt{3}i - i + \sqrt{3}}{4} = \frac{1 + \sqrt{3}}{4} + \frac{\sqrt{3}-1}{4}i$$

Modulus: $|z_4| = \frac{|1-i|}{|1-\sqrt{3}i|} = \frac{1}{\sqrt{2}}$. Phase: $\theta = \arctan\left(\frac{\sqrt{3}-1}{1+\sqrt{3}}\right) = \arctan(2 - \sqrt{3}) = \pi/12$.

$$z_4 = \frac{1 + \sqrt{3}}{4} + \frac{\sqrt{3}-1}{4}i = \frac{1}{\sqrt{2}} e^{i\pi/12}$$

(e) By Euler's formula:

$$z_5 = \cos \theta + i \sin \theta = e^{i\theta}$$

(f) Using $3 = e^{\ln 3}$:

$$z_6 = 3^i = (e^{\ln 3})^i = e^{i \ln 3} = \cos(\ln 3) + i \sin(\ln 3)$$

$$\boxed{z_6 = \cos(\ln 3) + i \sin(\ln 3) = e^{i \ln 3}}$$

(g) First simplify the argument:

$$\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i-1}{2} = i = e^{i\pi/2}$$

Therefore:

$$z_7 = \ln(i) = \ln(e^{i\pi/2}) = \frac{\pi}{2}i$$

$$\boxed{z_7 = \frac{\pi}{2}i = \frac{\pi}{2}e^{i\pi/2}}$$

2. Solve for x and y .

Rearranging and combining over a common denominator ($6 + 7i$):

$$\frac{2}{x+iy} = 2+9i - \frac{2+3i}{6+7i} = \frac{(2+9i)(6+7i) - (2+3i)}{6+7i} = \frac{-53+65i}{6+7i}$$

Inverting both sides and rationalizing:

$$x+iy = \frac{2(6+7i)}{-53+65i} = \frac{2(6+7i)(-53-65i)}{53^2+65^2} = \frac{2(137-761i)}{7034} = \frac{137-761i}{3517}$$

Therefore:

$$\boxed{x = \frac{137}{3517}, \quad y = -\frac{761}{3517}}$$

3. Sum, product, and quotient.

Using results from Problem 1:

$$z_1 + z_3 = (1-i) + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \boxed{\frac{1}{2} + \frac{\sqrt{3}-2}{2}i}$$

$$z_1 \cdot z_3 = |z_1||z_3|e^{i(\theta_1+\theta_3)} = \sqrt{2} \cdot 1 \cdot e^{i(-\pi/4+2\pi/3)} = \boxed{\sqrt{2}e^{i \cdot 5\pi/12}}$$

$$z_1/z_3 = \frac{|z_1|}{|z_3|}e^{i(\theta_1-\theta_3)} = \sqrt{2}e^{i(-\pi/4-2\pi/3)} = \boxed{\sqrt{2}e^{-i \cdot 11\pi/12}}$$

4. RLC circuit impedance.

(a) In a RLC circuit, according to Kirchhoff's law:

$$\Delta U_R + \Delta U_L + \Delta U_C = V(t) = V_0 e^{i\omega t}$$

$$\text{For a resistor: } \Delta U_R = I(t) \cdot R = R \frac{dQ(t)}{dt}$$

$$\text{For an inductor: } \Delta U_L = L \frac{dI(t)}{dt} = L \frac{d^2Q(t)}{dt^2}$$

$$\text{For a capacitor: } \Delta U_C = \frac{Q(t)}{C}$$

In a (co)sinusoidally varying circuit, the charge must have the form $Q(t) = Q_0 e^{i\omega t}$, and plugging into the equation:

$$Q(t) = \frac{V_0 e^{i\omega t}}{-\omega^2 L + i\omega R + 1/C}$$

The current is:

$$I(t) = \frac{dQ(t)}{dt} = \frac{V_0 e^{i\omega t}}{i\omega L + R + \frac{1}{i\omega C}}$$

i.e.,

$$Z = R + i\omega L + \frac{1}{i\omega C}$$

(b) At angular frequency $\omega = 2\pi \times 60/\text{s}$:

$$|Z(\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \approx 482 \Omega$$

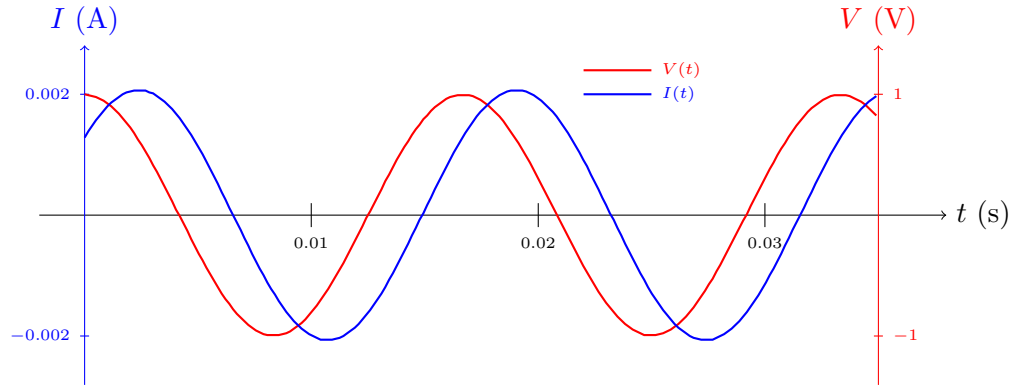
$$V_0/|Z(\omega)| \approx 0.00208 \text{ A}$$

The phase of the impedance:

$$\phi_{Z(\omega)} = \arctan\left(\frac{\omega L - 1/(\omega C)}{R}\right) \approx 0.899$$

The current is $I(t) = \frac{V_0}{|Z(\omega)|} e^{i(\omega t - \phi_Z)}$, so with $V_0 = 1 \text{ V}$ the real current response is:

$$I(t) = \frac{V_0}{|Z|} \cos(\omega t - \phi_Z) \approx 0.00208 \text{ A} \cos(120\pi t/\text{s} - 0.899)$$

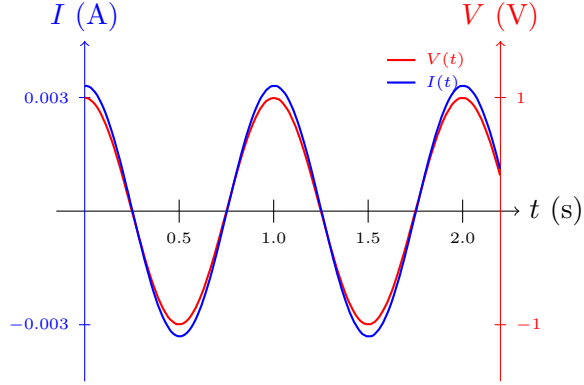


(c) If $\omega = 2\pi/\text{s}$:

$$|Z(\omega)| \approx 300.06 \Omega, \quad V_0/|Z(\omega)| \approx 0.00333 \text{ A}$$

$$\phi_{Z(\omega)} \approx 0.0204$$

$$I(t) \approx 0.00333 \text{ A} \cos(2\pi t/\text{s} - 0.0204)$$



5. Complex number identities.

(a) Let $z = x + iy$. Then $|z| = \sqrt{x^2 + y^2}$.

$$\operatorname{Re} z = x \leq \sqrt{x^2 + y^2} = |z|$$

$$\operatorname{Im} z = y \leq \sqrt{x^2 + y^2} = |z|$$

(b)

$$|zw| = \sqrt{(zw)(zw)^*} = \sqrt{(zz^*)(ww^*)} = \sqrt{|z|^2|w|^2} = |z||w|$$

(c)

$$\begin{aligned} |z + w|^2 &= (z + w)(z + w)^* \\ &= zz^* + ww^* + zw^* + z^*w \\ &= |z|^2 + |w|^2 + 2\operatorname{Re}(zw^*) \end{aligned}$$

6. Trigonometric addition formulas.

Expanding $e^{iA}e^{iB}$ using Euler's formula:

$$\begin{aligned} e^{iA}e^{iB} &= (\cos A + i \sin A)(\cos B + i \sin B) \\ &= \cos A \cos B - \sin A \sin B + i[\cos A \sin B + \sin A \cos B] \end{aligned}$$

On the other hand:

$$e^{i(A+B)} = \cos(A+B) + i \sin(A+B)$$

Since $e^{iA}e^{iB} = e^{i(A+B)}$, matching the real and imaginary parts:

$$\boxed{\cos(A+B) = \cos A \cos B - \sin A \sin B}$$

$$\boxed{\sin(A+B) = \cos A \sin B + \sin A \cos B}$$