

Problem Set 11: Eigensystems and Differential Equations

1. Eigenvalues and eigenvectors.

(a) Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}: \det(\sigma_x - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \boxed{\lambda = \pm 1}$$

$$|v_x^{(+)}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |v_x^{(-)}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}: \det(\sigma_y - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \boxed{\lambda = \pm 1}$$

$$|v_y^{(+)}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, |v_y^{(-)}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}: \det(\sigma_z - \lambda I) = (1 - \lambda)(-1 - \lambda) = 0 \Rightarrow \boxed{\lambda = \pm 1}$$

$$|v_z^{(+)}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |v_z^{(-)}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b) Identity matrix I :

$$\det(I - \lambda I) = (1 - \lambda)^2 = 0 \Rightarrow \boxed{\lambda = 1} \text{ (degenerate)}$$

Any vector is an eigenvector. Choose orthogonal: $|v_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |v_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(c) Zero matrix:

$$\det(0 - \lambda I) = \lambda^2 = 0 \Rightarrow \boxed{\lambda = 0} \text{ (degenerate)}$$

Any vector is an eigenvector. Choose: $|v_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |v_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$(d) D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 7 \end{pmatrix}:$$

Diagonal matrix, so eigenvalues are on diagonal: $\boxed{\lambda_1 = 1, \lambda_2 = \sqrt{5}, \lambda_3 = 7}$

$$|v_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |v_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |v_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(e) B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 1 \end{pmatrix}:$$

$$\det(B - \lambda I) = (2 - \lambda)[(1 - \lambda)^2 - 1] = (2 - \lambda)\lambda(\lambda - 2) = -\lambda(\lambda - 2)^2$$

Setting equal to zero: $\boxed{\lambda_1 = 2 \text{ (degenerate)}, \lambda_2 = 0}$

For $\lambda = 2$: $|v_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$

For $\lambda = 0$: $|v_3\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$

2. Eigenvalues of A^{-1} .

Given $Av_i = \lambda_i v_i$.

Multiply both sides by A^{-1} :

$$A^{-1}Av_i = \lambda_i A^{-1}v_i$$

$$v_i = \lambda_i A^{-1}v_i$$

Dividing by λ_i (assuming $\lambda_i \neq 0$):

$$\boxed{\frac{1}{\lambda_i} v_i = A^{-1} v_i}$$

Thus A^{-1} has the same eigenvectors v_i with eigenvalues $1/\lambda_i$. $\square$

(Note that this requires A to be invertible, which means none of the λ_i can be zero in the first place.)

3. Orthogonal matrices.

(a) If A and B are orthogonal ($A^{-1} = A^T$, $B^{-1} = B^T$), then:

$$(AB)^{-1} = B^{-1}A^{-1} = B^T A^T = (AB)^T$$

Thus AB is orthogonal. $\square$

(b) Since A is orthogonal: $A^{-1} = A^T$.

Taking the transpose: $(A^{-1})^T = (A^T)^T = A$.

But also $(A^{-1})^{-1} = A$.

So $(A^{-1})^T = (A^{-1})^{-1}$, which means A^{-1} is orthogonal. $\square$

(c) Using $AA^T = I$:

$$\det(AA^T) = \det(A) \cdot \det(A^T) = (\det A)^2$$

On the other hand:

$$\det(AA^T) = \det(AA^{-1}) = \det(I) = 1$$

Therefore $(\det A)^2 = 1$, so $\boxed{\det A = \pm 1}$. $\square$

(d) Assume $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Since $\det A = +1$ for $SO(2)$, the inverse simplifies:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Setting this equal to

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

and comparing entries gives $d = a$ and $c = -b$, so A has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a^2 + b^2 = 1$ (from $\det A = a^2 + b^2 = 1$).

Parametrize: $a = \cos \theta$, $b = \sin \theta$:

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

Geometric interpretation: A rotates a vector by angle θ counterclockwise about the origin.

(e) An element of $O(2)$ with $\det A = -1$ is, for example,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Geometric interpretation: this is a reflection across the x -axis. (More generally, any $\det = -1$ element of $O(2)$ is a reflection across some line through the origin.)

4. Hermitian operator $\hat{p} = -i \frac{d}{dx}$.

(a) To show $\hat{p}$ is Hermitian, we need $\langle f | \hat{p} g \rangle = \langle \hat{p} f | g \rangle$. Integrate by parts:

$$\begin{aligned} \langle f | \hat{p} g \rangle &= \int_0^1 dx f^*(x) \left(-i \frac{dg}{dx} \right) \\ &= -i \int_0^1 dx \frac{d}{dx} [f^*(x) g(x)] + i \int_0^1 dx \frac{df^*}{dx} g(x) \\ &= -i [f^*(x) g(x)]_0^1 + \int_0^1 dx \left(-i \frac{df}{dx} \right)^* g(x) \end{aligned}$$

The boundary term vanishes due to periodic boundary conditions ($f(0) = f(1)$, $g(0) = g(1)$), so

$$\langle f | \hat{p} g \rangle = \int_0^1 dx (\hat{p} f)^* g(x) = \boxed{\langle \hat{p} f | g \rangle} \quad \checkmark$$

(b) Solve $\hat{p} f = \lambda f$:

$$-i \frac{df}{dx} = \lambda f \quad \Rightarrow \quad \frac{df}{f} = i \lambda dx$$

Integrating: $\ln f = i \lambda x + C$, so $f(x) = C_n e^{i \lambda x}$.

Applying the periodic BC $f(1) = f(0)$ gives $e^{i \lambda} = 1$, so $\lambda = 2\pi n$ for $n \in \mathbb{Z}$.

Spectrum: $\boxed{\lambda_n = 2\pi n, \quad n \in \mathbb{Z}}$

Normalizing: $|C_n|^2 \int_0^1 dx e^{-i \lambda_n x} e^{i \lambda_n x} = |C_n|^2 = 1$, so $C_n = 1$ (up to a phase).

Eigenfunctions: $\boxed{f_n(x) = e^{i 2\pi n x}}$

5. Boat drag force.

The equation is $m \frac{dv}{dt} = -kv^n$.

- (a) **Linear:** The equation is linear if v appears to the first power only.

For $n = 0$: $m\dot{v} + k = 0$ (linear, inhomogeneous).

For $n = 1$: $m\dot{v} + kv = 0$ (linear, homogeneous).

For $|n| \geq 2$: nonlinear.

$$\boxed{n = 0 \text{ or } n = 1}$$

- (b) **Homogeneous:** The equation is homogeneous if there is no inhomogeneous (constant) term.

$$\boxed{n = 1} \text{ (for } n = 0 \text{ there is a constant term } -k/m\text{)}.$$

- (c) **Velocity as a function of time.**

Separate variables:

$$\int_{v_0}^v \frac{dv'}{(v')^n} = -\frac{k}{m} \int_0^t dt'$$

For $n \neq 1$:

$$\frac{v^{1-n}}{1-n} - \frac{v_0^{1-n}}{1-n} = -\frac{k}{m}t$$

Solving for $v(t)$:

$$\boxed{v(t) = \left[v_0^{1-n} - (1-n)\frac{k}{m}t \right]^{\frac{1}{1-n}}} \quad (n \neq 1)$$

For $n = 1$:

$$\boxed{v(t) = v_0 e^{-kt/m}}$$

6. Third-order ODE.

$y''' - 2y'' + y' - 2y = 0$ is linear and homogeneous with constant coefficients.

Try the exponential ansatz $y = e^{cx}$, giving $y' = cy$, $y'' = c^2y$, $y''' = c^3y$:

$$c^3 - 2c^2 + c - 2 = 0$$

Factor by grouping:

$$c^3 - 2c^2 + c - 2 = c^2(c - 2) + (c - 2) = (c^2 + 1)(c - 2) = (c + i)(c - i)(c - 2) = 0$$

Roots: $c_1 = 2$, $c_2 = i$, $c_3 = -i$.

General solution:

$$\boxed{y(x) = C_1 e^{2x} + C_2 e^{ix} + C_3 e^{-ix}}$$

Or equivalently, using real combinations of the complex exponentials,

$$y(x) = C_1 e^{2x} + A \cos x + B \sin x.$$

7. Radioactive decay series.

(a) Solve $\frac{dN_1}{dt} = -\lambda_1 N_1$:

$$\int_{N_0}^{N_1} \frac{dN'_1}{N'_1} = -\lambda_1 \int_0^t dt' \Rightarrow \ln\left(\frac{N_1}{N_0}\right) = -\lambda_1 t$$

$$\boxed{N_1(t) = N_0 e^{-\lambda_1 t}}$$

(b) Plug into the second equation:

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_0 e^{-\lambda_1 t}$$

This is linear and inhomogeneous.

Homogeneous solution: $\frac{dN_2}{dt} = -\lambda_2 N_2 \Rightarrow N_{2,h} = C e^{-\lambda_2 t}$.

Particular solution. Guess $N_p = N_{p,0} e^{-\lambda_1 t}$:

$$-\lambda_1 N_{p,0} e^{-\lambda_1 t} + \lambda_2 N_{p,0} e^{-\lambda_1 t} = \lambda_1 N_0 e^{-\lambda_1 t}$$

$$(\lambda_2 - \lambda_1) N_{p,0} = \lambda_1 N_0 \Rightarrow N_{p,0} = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}$$

General solution:

$$N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} e^{-\lambda_1 t} + C e^{-\lambda_2 t}$$

Apply $N_2(0) = 0$:

$$0 = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} + C \Rightarrow C = -\frac{\lambda_1 N_0}{\lambda_2 - \lambda_1}$$

Final answer:

$$\boxed{N_2(t) = \frac{\lambda_1 N_0}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right)}$$