

Problem Set 10: Determinants and More Matrices

1. Determinants.

$$(a) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 0 + (-1) = \boxed{-1}$$

$$(b) \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = \boxed{-5}$$

(c) Expand along the first row. Only the (1,2) entry is nonzero:

$$\begin{vmatrix} 0 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 3 & 0 \\ 0 & 3 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \end{vmatrix} = -\sqrt{2} \begin{vmatrix} \sqrt{2} & 3 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{vmatrix}$$

Now expand the 3×3 determinant along its first row:

$$= -\sqrt{2} \left[\sqrt{2} \begin{vmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{vmatrix} - 3 \begin{vmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{vmatrix} \right] = -\sqrt{2} [\sqrt{2}(0 - 2) - 3(0)] = -\sqrt{2}(-2\sqrt{2}) = \boxed{4}$$

2. Three vectors.

$$\vec{u} = \hat{e}_x + \hat{e}_y, \vec{v} = \hat{e}_y + \hat{e}_z, \vec{w} = \hat{e}_x + \hat{e}_z$$

(a) Length:

$$|\vec{u}| = \sqrt{1+1} = \sqrt{2}, \quad |\vec{v}| = \sqrt{2}, \quad |\vec{w}| = \sqrt{2}$$

All vectors have length $\boxed{\sqrt{2}}$.

(b) Angle between vectors:

$$\vec{u} \cdot \vec{v} = 1, \quad \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{1}{2}$$

$$\text{So } \theta_{uv} = \theta_{uw} = \theta_{vw} = \boxed{\pi/3}$$

(c) Volume of parallelepiped:

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = \left| \det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right| = |1(1) - 0 + 1(1)| = \boxed{2}$$

3. LU decomposition.

$A = LU$ where L is lower triangular and U is upper triangular.

$$\det(A) = \det(L) \cdot \det(U)$$

For triangular matrices, the determinant is the product of diagonal elements:

$$\begin{aligned}\det(L) &= 2 \times 7 \times 3 \times 1 = 42 \\ \det(U) &= 1 \times (-1) \times 4 \times 2 = -8\end{aligned}$$

Therefore:

$$\det(A) = 42 \times (-8) = \boxed{-336}$$

4. Trace of angular momentum.

First, note the property: $\text{Tr}(AB) = \text{Tr}(BA)$.

From the commutation relation: $[L_j, L_k] = i\epsilon_{jkl}L_l$

Taking the trace:

$$\text{Tr}([L_j, L_k]) = \text{Tr}(L_j L_k) - \text{Tr}(L_k L_j) = 0$$

So: $i\epsilon_{jkl}\text{Tr}(L_l) = 0$ for all j, k .

Taking $j = x, k = y$: $i\epsilon_{xyz}\text{Tr}(L_z) = i\text{Tr}(L_z) = 0 \Rightarrow \boxed{\text{Tr}(L_z) = 0}$

Similarly for L_x and L_y . $\square$

5. Hermitian matrices.

Recall that M is Hermitian if $M^\dagger = M$, where $M^\dagger = \overline{M^T}$.

- (a) $(A + B)^\dagger = \overline{(A + B)^T} = \overline{A^T} + \overline{B^T} = A^\dagger + B^\dagger = A + B \quad \checkmark$
- (b) $(AB + BA)^\dagger = \overline{(AB + BA)^T} = \overline{B^T A^T} + \overline{A^T B^T} = B^\dagger A^\dagger + A^\dagger B^\dagger = BA + AB \quad \checkmark$
- (c) $(i[A, B])^\dagger = -i\overline{(AB - BA)^T} = -i(\overline{B^T A^T} - \overline{A^T B^T}) = -i(B^\dagger A^\dagger - A^\dagger B^\dagger)$
 $= -i(BA - AB) = i(AB - BA) = i[A, B] \quad \checkmark$

6. Linear independence equivalence.

We need to show the two definitions are equivalent:

Def 1: No vector in the set can be written as a linear combination of the others.

Def 2: $\sum_i a_i \vec{v}_i = \vec{0}$ implies all $a_i = 0$.

Def 1 $\Rightarrow$ Def 2: Suppose $\sum_i a_i \vec{v}_i = \vec{0}$ and some coefficient $a_k \neq 0$. Then we can solve for $\vec{v}_k$:

$$\vec{v}_k = -\frac{1}{a_k} \sum_{i \neq k} a_i \vec{v}_i$$

which writes $\vec{v}_k$ as a linear combination of the other vectors, contradicting Def 1. So all $a_i = 0$.

Def 2 $\Rightarrow$ Def 1: Suppose some vector can be written as a combination of the others, say $\vec{v}_1 = c_2 \vec{v}_2 + c_3 \vec{v}_3 + \dots + c_n \vec{v}_n$. Then:

$$1 \cdot \vec{v}_1 - c_2 \vec{v}_2 - c_3 \vec{v}_3 - \dots - c_n \vec{v}_n = \vec{0}$$

with the coefficient of $\vec{v}_1$ equal to $1 \neq 0$, contradicting Def 2. $\square$

7. Columns and determinant.

Let $\vec{v}_1, \dots, \vec{v}_n$ be the columns of B , so $B = (\vec{v}_1 \mid \vec{v}_2 \mid \dots \mid \vec{v}_n)$.

If the columns are linearly dependent, then (without loss of generality) one column can be written as a linear combination of the others:

$$\vec{v}_1 = a_2\vec{v}_2 + a_3\vec{v}_3 + \dots + a_n\vec{v}_n$$

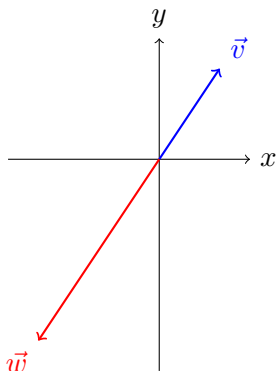
Now use the fact that the determinant is linear in each column. Replacing the first column:

$$\det B = \det(a_2\vec{v}_2 + \dots + a_n\vec{v}_n, \vec{v}_2, \dots, \vec{v}_n) = \sum_{k=2}^n a_k \det(\vec{v}_k, \vec{v}_2, \dots, \vec{v}_n)$$

Each term in the sum has two identical columns ($\vec{v}_k$ appears twice), so each determinant vanishes. Therefore $\det B = 0$. $\square$

8. Span and basis.

(a) $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \vec{w} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$

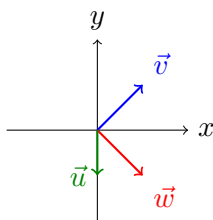


$\vec{w} = -2\vec{v}$, so the vectors are **linearly dependent**.

They span only a line through the origin, so they do **not span** $\mathbb{R}^2$.

Not a basis for $\mathbb{R}^2$.

(b) $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \vec{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

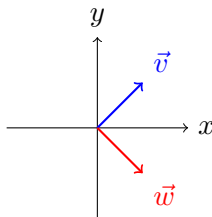


Note: $\vec{v} = \vec{w} - 2\vec{u}$, so the vectors are **linearly dependent**.

They **span** $\mathbb{R}^2$ (any two of them form a basis).

Not a basis (too many vectors).

(c) $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



$\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0$, so the vectors are **linearly independent**.

They **span** $\mathbb{R}^2$.

Forms a basis for $\mathbb{R}^2$.

9. Polynomial space P_n .

(a) P_n forms a vector space over $\mathbb{C}$. We verify the axioms:

- i. Commutative: $p(x) + q(x) = q(x) + p(x)$ ✓
- ii. Associative: $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$ ✓
- iii. Additive identity: The zero polynomial 0 satisfies $p(x) + 0 = p(x)$ ✓
- iv. Additive inverse: $-p(x)$ satisfies $p(x) + (-p(x)) = 0$ ✓
- v. Associative scalar mult: $\lambda(\mu p(x)) = (\lambda\mu)p(x)$ ✓
- vi. Distributive: $\lambda(p + q) = \lambda p + \lambda q$ ✓
- vii. Distributive: $(\lambda + \mu)p = \lambda p + \mu p$ ✓
- viii. Unit multiplication: $1 \cdot p(x) = p(x)$ ✓

(b) Any polynomial $p(x) = a_0 + a_1x + \dots + a_nx^n$ can be written as a linear combination of $S = \{1, x, x^2, \dots, x^n\}$ with coordinates $(a_0, a_1, \dots, a_n)$.

The monomials in S are linearly independent (no monomial can be written as a combination of others), and any polynomial can be written as a linear combination of them. Thus S is a basis.

(c) The polynomial $p_n(x) = a_0 + a_1x + \dots + a_nx^n$ is determined by $n+1$ complex coefficients.

$$\boxed{\dim P_n = n + 1}$$

(d) For P_3 , represent $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ as a column vector $(a_0, a_1, a_2, a_3)^T$.

$$\frac{d}{dx}p(x) = a_1 + 2a_2x + 3a_3x^2 \rightarrow (a_1, 2a_2, 3a_3, 0)^T$$

The derivative matrix is:

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(e) D is upper triangular with all zeros on the diagonal.

$$\boxed{\det D = 0}$$

(f) Since $\det D = 0$, D is not invertible, so $\frac{d}{dx}$ is **not invertible** on P_n .

This makes sense: multiple polynomials have the same derivative (they differ by a constant), so the derivative map is not one-to-one.

10. Gram-Schmidt procedure.

$$\begin{aligned}
 \text{(a)} \quad \vec{v}_1 &= 3\hat{x} + 4\hat{y}, \quad \vec{v}_2 = 2\hat{x} - 5\hat{y} \\
 \hat{e}_1 &= \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{1}{5}(3\hat{x} + 4\hat{y}) = \frac{3}{5}\hat{x} + \frac{4}{5}\hat{y} \\
 \vec{v}_{2\perp} &= \vec{v}_2 - \hat{e}_1(\hat{e}_1 \cdot \vec{v}_2) = (2\hat{x} - 5\hat{y}) - \left(\frac{3}{5}\hat{x} + \frac{4}{5}\hat{y}\right) \left(\frac{6}{5} - \frac{20}{5}\right) \\
 &= (2\hat{x} - 5\hat{y}) + \frac{14}{5} \left(\frac{3}{5}\hat{x} + \frac{4}{5}\hat{y}\right) = \frac{92}{25}\hat{x} - \frac{69}{25}\hat{y} \\
 |\vec{v}_{2\perp}| &= \frac{23}{5}, \text{ so } \hat{e}_2 = \frac{\vec{v}_{2\perp}}{|\vec{v}_{2\perp}|} = \frac{4}{5}\hat{x} - \frac{3}{5}\hat{y}
 \end{aligned}$$

$$\hat{e}_1 = \frac{3}{5}\hat{x} + \frac{4}{5}\hat{y}, \quad \hat{e}_2 = \frac{4}{5}\hat{x} - \frac{3}{5}\hat{y}$$

$$\begin{aligned}
 \text{(b)} \quad |v_1\rangle &= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad |v_2\rangle = \begin{pmatrix} 0 \\ 2 \\ i \end{pmatrix}, \quad |v_3\rangle = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \\
 |e_1\rangle &= \frac{|v_1\rangle}{\sqrt{\langle v_1|v_1\rangle}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
 |v_{2\perp}\rangle &= |v_2\rangle - |e_1\rangle\langle e_1|v_2\rangle = \begin{pmatrix} 0 \\ 2 \\ i \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (0) = \begin{pmatrix} 0 \\ 2 \\ i \end{pmatrix} \\
 \langle v_{2\perp}|v_{2\perp}\rangle &= 4 + 1 = 5, \text{ so } |e_2\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 2 \\ i \end{pmatrix} \\
 \langle e_1|v_3\rangle &= 0, \quad \langle e_2|v_3\rangle = \frac{1}{\sqrt{5}}(2 \cdot 1 + (-i) \cdot 3) = \frac{2-3i}{\sqrt{5}} \\
 |v_{3\perp}\rangle &= |v_3\rangle - |e_1\rangle\langle e_1|v_3\rangle - |e_2\rangle\langle e_2|v_3\rangle \\
 &= \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 2 \\ i \end{pmatrix} \cdot \frac{2-3i}{\sqrt{5}} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - \frac{2-3i}{5} \begin{pmatrix} 0 \\ 2 \\ i \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 1 - \frac{4-6i}{5} \\ 3 - \frac{2i+3}{5} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1+6i}{5} \\ \frac{12-2i}{5} \end{pmatrix} \\
 \langle v_{3\perp}|v_{3\perp}\rangle &= \frac{1}{25}(1 + 36 + 144 + 4) = \frac{185}{25} = \frac{37}{5}
 \end{aligned}$$

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |e_2\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 2 \\ i \end{pmatrix}, \quad |e_3\rangle = \frac{1}{\sqrt{185}} \begin{pmatrix} 0 \\ 1+6i \\ 12-2i \end{pmatrix}$$

$$\text{(c)} \quad p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2 \text{ with } \langle p|q\rangle = \int_{-1}^1 dx p^*(x)q(x)$$

A note on inner products of functions: The inner product $\langle p|q\rangle$ plays the same role for functions that the dot product plays for vectors. Two functions are orthogonal if $\langle p|q\rangle = 0$, and a function's norm (its "length") is $\|p\| = \sqrt{\langle p|p\rangle}$. An orthonormal basis of functions $\{e_i(x)\}$ therefore satisfies $\langle e_i|e_j\rangle = \delta_{ij}$, just like an orthonormal basis of vectors.

$$\langle p_0|p_0\rangle = \int_{-1}^1 dx = 2, \text{ so } e_0 = \frac{1}{\sqrt{2}}$$

$$\begin{aligned}
\langle e_0|p_1\rangle &= \frac{1}{\sqrt{2}} \int_{-1}^1 x \, dx = 0 \\
p_{1\perp} &= p_1 = x, \quad \langle p_{1\perp}|p_{1\perp}\rangle = \int_{-1}^1 x^2 \, dx = \frac{2}{3} \\
e_1 &= \sqrt{\frac{3}{2}} x \\
\langle e_0|p_2\rangle &= \frac{1}{\sqrt{2}} \int_{-1}^1 x^2 \, dx = \frac{1}{\sqrt{2}} \cdot \frac{2}{3} = \frac{\sqrt{2}}{3} \\
\langle e_1|p_2\rangle &= \sqrt{\frac{3}{2}} \int_{-1}^1 x^3 \, dx = 0 \\
p_{2\perp} &= x^2 - \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{3} = x^2 - \frac{1}{3} \\
\langle p_{2\perp}|p_{2\perp}\rangle &= \int_{-1}^1 (x^2 - \frac{1}{3})^2 \, dx = \frac{8}{45} \\
e_2 &= \sqrt{\frac{45}{8}} (x^2 - \frac{1}{3}) = \frac{3}{2} \sqrt{\frac{5}{2}} (x^2 - \frac{1}{3})
\end{aligned}$$

$$\boxed{e_0 = \frac{1}{\sqrt{2}}, \quad e_1 = \sqrt{\frac{3}{2}} x, \quad e_2 = \frac{3}{2} \sqrt{\frac{5}{2}} \left(x^2 - \frac{1}{3} \right)}$$

(These are proportional to the Legendre polynomials P_0 , P_1 , P_2 .)