

1. $\mathbb{N}$ denotes the natural numbers including 0, $\mathbb{Z}$ the integers.

(a) **Find the composite map $C = B \circ A$.**

The composite map $C : \mathbb{Z} \rightarrow \mathbb{N}$ is defined by

$$C(n) = (B \circ A)(n) = B(A(n)) = B(n + 1) = |n + 1|$$

So:

$$C(n) = |n + 1| = \begin{cases} n + 1 & \text{if } n \geq -1 \\ -(n + 1) & \text{if } n < -1 \end{cases}$$

Domain: $\mathbb{Z}$

Image: $\mathbb{N}$

Action on n : $n \mapsto |n + 1|$

(b) **Which maps are injective, surjective, bijective?**

Map $A : n \mapsto n + 1$ (shift by 1)

- *Injective?* Yes — different inputs give different outputs since if $n + 1 = m + 1$ then $n = m$.
- *Surjective?* Yes — every integer ℓ is hit: just take $n = \ell - 1$.
- *Bijjective?* Yes (both injective and surjective). This is just a shift, which is clearly invertible.

Map $B : n \mapsto |n|$ (absolute value)

- *Injective?* No — for example, $B(1) = B(-1) = 1$, so two different inputs give the same output.
- *Surjective?* Yes — every natural number $\ell \geq 0$ is hit: just take $n = \ell$.
- *Bijjective?* No (fails injectivity).

Map $C : n \mapsto |n + 1|$ (shift then absolute value)

- *Injective?* No — for example, $C(0) = |1| = 1$ and $C(-2) = |-1| = 1$.
- *Surjective?* Yes — every $\ell \in \mathbb{N}$ is hit: take $n = \ell - 1$.
- *Bijjective?* No (fails injectivity).

Summary:

	Injective	Surjective	Bijjective
A	✓	✓	✓
B	×	✓	×
C	×	✓	×

2. **Compute the first derivative analytically:**

(a) $f(x) = 2^x$

Let $y = 2^x$, so $\ln y = x \ln 2$.

Differentiating: $\frac{1}{y} \frac{dy}{dx} = \ln 2$

Therefore:

$$\boxed{\frac{df}{dx} = y \ln 2 = 2^x \ln 2}$$

Alternative: Write $2^x = e^{x \ln 2}$, then $\frac{d}{dx} e^{x \ln 2} = e^{x \ln 2} \cdot \ln 2 = 2^x \ln 2$.

(b) $f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$

$$\frac{df}{dx} = \frac{1}{2} (e^x - (-1)e^{-x}) = \frac{1}{2} (e^x + e^{-x})$$

$$\boxed{\frac{df}{dx} = \cosh x}$$

(c) $f(x) = \tanh x = \frac{\sinh x}{\cosh x}$

Using the quotient rule:

$$\frac{df}{dx} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

Using the identity $\cosh^2 x - \sinh^2 x = 1$:

$$\boxed{\frac{df}{dx} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x}$$

(d) $f(x) = x \ln x$

Using the product rule:

$$\frac{df}{dx} = \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

$$\boxed{\frac{df}{dx} = \ln x + 1}$$

(e) $f(x) = \cos(x^3 + 1)$

Using the chain rule with $u = x^3 + 1$:

$$\frac{df}{dx} = -\sin(x^3 + 1) \cdot \frac{d}{dx}(x^3 + 1) = -\sin(x^3 + 1) \cdot 3x^2$$

$$\boxed{\frac{df}{dx} = -3x^2 \sin(x^3 + 1)}$$

3. Sketch the following functions:

(a) $f(x) = \sinh x = \frac{1}{2}(e^x - e^{-x})$

Domain: $\mathbb{R}$ (all real numbers)

Zeros: $f(0) = 0$, so $x = 0$ is the only zero. The slope there is $f'(0) = \cosh(0) = 1$.

Limiting behavior:

- As $x \rightarrow \infty$: $f(x) \approx \frac{1}{2}e^x \rightarrow \infty$
- As $x \rightarrow -\infty$: $f(x) \approx -\frac{1}{2}e^{-x} \rightarrow -\infty$

Symmetry: $\sinh(-x) = -\sinh(x)$, so f is an odd function.

Divergences: None for finite x .

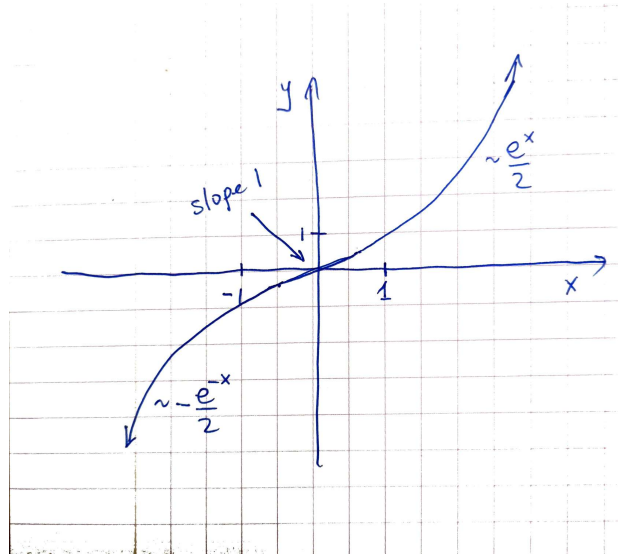


Figure 1: Graph of $f(x) = \sinh x$

(b) $f(x) = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Domain: $\mathbb{R}$ (denominator $\cosh x > 0$ always)

Zeros: $\tanh x = 0$ when $\sinh x = 0$, i.e., $x = 0$.

Limiting behavior:

- As $x \rightarrow \infty$: $\tanh x \approx \frac{e^x}{e^x} = 1 - 2e^{-2x} \rightarrow 1$
- As $x \rightarrow -\infty$: $\tanh x \approx \frac{-e^{-x}}{e^{-x}} = -1 + 2e^{2x} \rightarrow -1$

Near origin: $\tanh x \approx \frac{1+x - (1-x)}{1+x + (1-x)} = \frac{2x}{2} = x$

Symmetry: $\tanh(-x) = -\tanh(x)$, so f is an odd function.

Divergences: None for finite x .

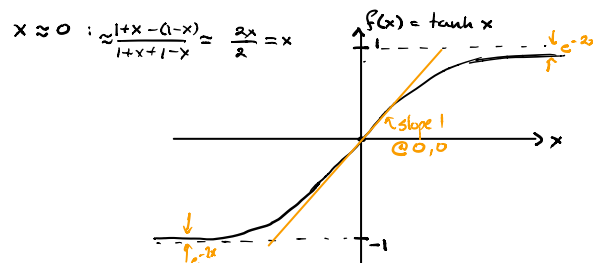


Figure 2: Graph of $f(x) = \tanh x$, with horizontal asymptotes at $y = \pm 1$

(c) $f(x) = x \ln x$

Domain: $(0, \infty)$ since $\ln x$ requires $x > 0$.

Zeros: $x \ln x = 0$ when $x = 1$ (since $\ln 1 = 0$).

Limiting behavior:

- As $x \rightarrow 0^+$: Use L'Hôpital's rule on $\frac{\ln x}{1/x}$:

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0$$

- As $x \rightarrow \infty$: $x \ln x \rightarrow \infty$

Critical points: $f'(x) = 1 + \ln x = 0 \implies x = e^{-1} = 1/e$

At $x = 1/e$: $f(1/e) = \frac{1}{e} \ln(1/e) = -\frac{1}{e} \approx -0.368$ (minimum)

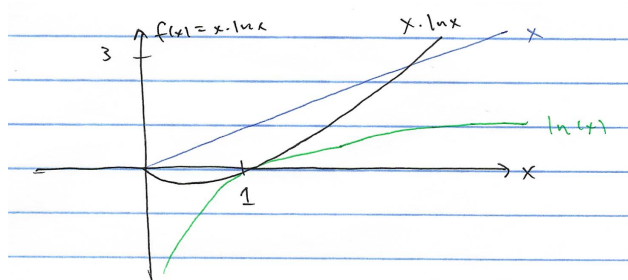


Figure 3: Graph of $f(x) = x \ln x$, with minimum at $(1/e, -1/e)$

(d) $f(x) = \frac{x^2 - 4}{x^2 - 1}$

Symmetry: $f(-x) = f(x)$, so f is an even function. We only need to analyze $x \geq 0$ and reflect symmetrically to negative x .

Domain: $\mathbb{R} \setminus \{-1, 1\}$ (denominator zero at $x = \pm 1$)

Zeros: $x^2 - 4 = 0 \implies x = \pm 2$

Divergences: Vertical asymptotes at $x = \pm 1$

At $x = 0$: $f(0) = \frac{-4}{-1} = 4$

Limiting behavior: As $x \rightarrow \pm\infty$:

$$f(x) = \frac{x^2 - 4}{x^2 - 1} = \frac{1 - 4/x^2}{1 - 1/x^2} \rightarrow 1$$

So $y = 1$ is a horizontal asymptote.

Behavior near $x = 1$: numerator $\rightarrow -3$, so $f \rightarrow +\infty$ from the left and $f \rightarrow -\infty$ from the right. The behavior near $x = -1$ follows by symmetry.

We can factor: $f(x) = \frac{(x+2)(x-2)}{(x+1)(x-1)}$

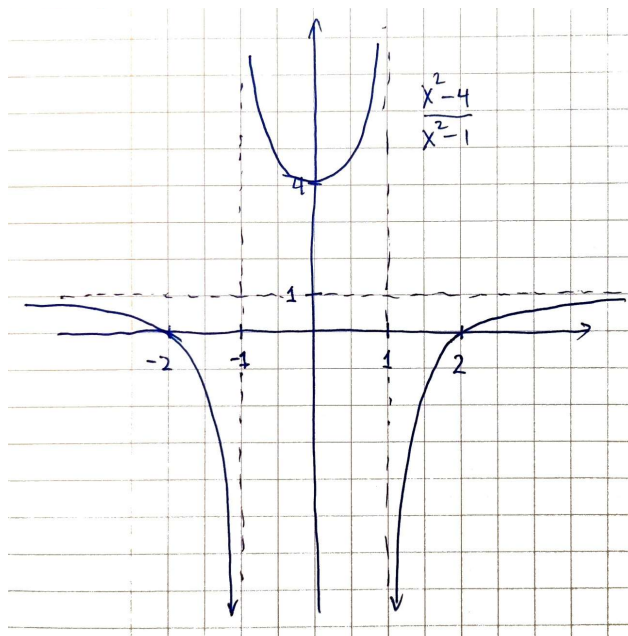


Figure 4: Graph of $f(x) = \frac{x^2 - 4}{x^2 - 1}$

4. Derivatives of inverse trigonometric functions.

Recall the inverse function rule: $\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))}$

(a) $f^{-1}(x) = \arcsin x$, for $x \in (-1, 1)$

Here $f(y) = \sin y$, so $f'(y) = \cos y$.

We need $\cos(\arcsin x)$. Using $\sin^2 \theta + \cos^2 \theta = 1$:

$$\cos(\arcsin x) = \sqrt{1 - \sin^2(\arcsin x)} = \sqrt{1 - x^2}$$

(positive since $\arcsin x \in (-\pi/2, \pi/2)$ where cosine is positive)

Therefore:

$$\boxed{\frac{d}{dx}(\arcsin x) = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - x^2}}}$$

Sketch: $\sin : [-\pi/2, \pi/2] \rightarrow [-1, 1]$ is bijective, so we can define $\arcsin : [-1, 1] \rightarrow [-\pi/2, \pi/2]$.

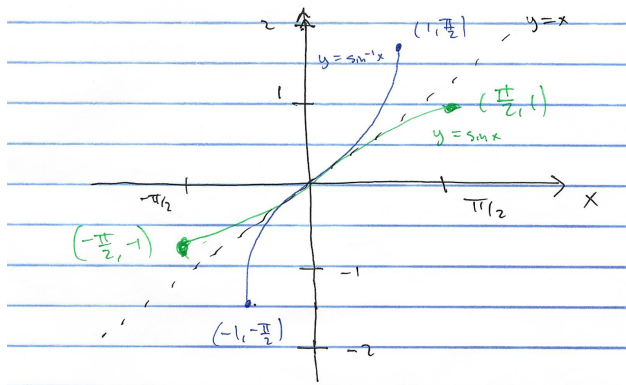


Figure 5: Graphs of $\sin x$ (restricted) and $\arcsin x$

(b) $f^{-1}(x) = \arctan x$

Here $f(y) = \tan y$, so $f'(y) = \sec^2 y = \frac{1}{\cos^2 y}$.

We need $\cos^2(\arctan x)$. Using $\tan^2 \theta + 1 = 1/\cos^2 \theta$:

$$\cos^2(\arctan x) = \frac{1}{1 + \tan^2(\arctan x)} = \frac{1}{1 + x^2}$$

Therefore:

$$\boxed{\frac{d}{dx}(\arctan x) = \cos^2(\arctan x) = \frac{1}{1 + x^2}}$$

Sketch: $\tan : (-\pi/2, \pi/2) \rightarrow \mathbb{R}$ is bijective, so we can define $\arctan : \mathbb{R} \rightarrow (-\pi/2, \pi/2)$.

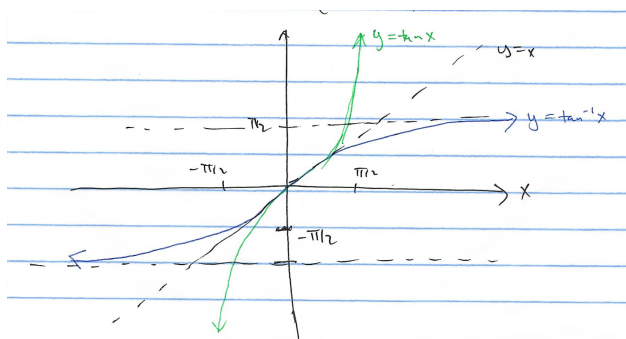


Figure 6: Graphs of $\tan x$ (restricted) and $\arctan x$

5. Taylor expansion of $f(x) = e^{x/(1-x)}$ near the origin.

Let $u = \frac{x}{1-x}$. Note that at $x = 0$, $u = 0$.

We need $f(0)$, $f'(0)$, and $f''(0)$.

$$f(x) = e^{x/(1-x)}$$

$$\text{At } x = 0: f(0) = e^0 = 1$$

For the derivatives, let's compute:

$$f'(x) = e^{x/(1-x)} \cdot \frac{d}{dx} \left(\frac{x}{1-x} \right) = e^{x/(1-x)} \cdot \frac{(1-x) + x}{(1-x)^2} = e^{x/(1-x)} \cdot \frac{1}{(1-x)^2}$$

At $x = 0$: $f'(0) = e^0 \cdot 1 = 1$

For $f''(x)$, using product rule:

$$\begin{aligned} f''(x) &= \frac{d}{dx} \left[e^{x/(1-x)} \right] \cdot \frac{1}{(1-x)^2} + e^{x/(1-x)} \cdot \frac{d}{dx} \left[\frac{1}{(1-x)^2} \right] \\ &= e^{x/(1-x)} \cdot \frac{1}{(1-x)^2} \cdot \frac{1}{(1-x)^2} + e^{x/(1-x)} \cdot \frac{2}{(1-x)^3} \\ &= e^{x/(1-x)} \left[\frac{1}{(1-x)^4} + \frac{2}{(1-x)^3} \right] \end{aligned}$$

At $x = 0$: $f''(0) = e^0(1 + 2) = 3$

Therefore, the Taylor expansion to second order is:

$$\boxed{f(x) = e^{x/(1-x)} \approx 1 + x + \frac{3}{2}x^2 + O(x^3)}$$

Alternative: Since $\frac{x}{1-x} = O(x)$, expanding $e^{(\cdot)}$ to second order in its argument is sufficient:

$$e^{x/(1-x)} \approx 1 + \frac{x}{1-x} + \frac{1}{2} \left(\frac{x}{1-x} \right)^2$$

Now $\frac{x}{1-x} = x(1 + x + \dots) = x + x^2 + O(x^3)$ and $\left(\frac{x}{1-x} \right)^2 = x^2 + O(x^3)$, so:

$$e^{x/(1-x)} \approx 1 + (x + x^2) + \frac{1}{2}x^2 + O(x^3) = 1 + x + \frac{3}{2}x^2 + O(x^3)$$

Qualitative features of $f(x) = e^{x/(1-x)}$:

- **Domain:** $\mathbb{R} \setminus \{1\}$
- At $x = 0$: $f(0) = 1$
- At $x = 1/2$: $f(1/2) = e^1 \approx 2.718$
- As $x \rightarrow 1^-$: $\frac{x}{1-x} \rightarrow +\infty$, so $f(x) \rightarrow \infty$
- As $x \rightarrow 1^+$: $\frac{x}{1-x} \rightarrow -\infty$, so $f(x) \rightarrow 0$
- As $x \rightarrow -\infty$: $\frac{x}{1-x} \rightarrow -1$, so $f(x) \rightarrow e^{-1} \approx 0.368$
- As $x \rightarrow +\infty$: $\frac{x}{1-x} \rightarrow -1$, so $f(x) \rightarrow e^{-1} \approx 0.368$

The second-order approximation $g(x) = 1 + x + \frac{3}{2}x^2$ is roughly good over $x \in [-1/2, 1/2]$, as expected since we expanded around $x = 0$. The approximation must clearly fail as $x \rightarrow 1$ where $f(x) \rightarrow \infty$, while the parabola remains finite.

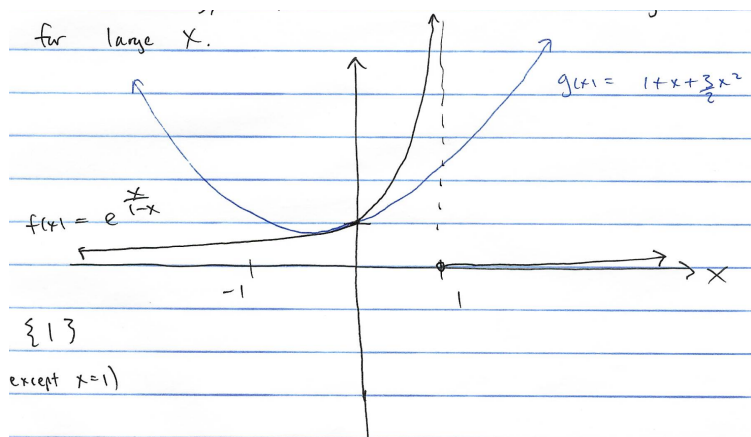


Figure 7: Graph of $f(x) = e^{x/(1-x)}$ and its second-order Taylor approximation $g(x) = 1 + x + \frac{3}{2}x^2 = \frac{3}{2}(x + \frac{1}{3})^2 + \frac{5}{6}$, a parabola with minimum at $(-\frac{1}{3}, \frac{5}{6})$.

6. Evaluate the following limits:

(a) $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

Using Taylor expansion: $\sin x = x - \frac{x^3}{6} + \dots$

$$\frac{\sin x}{x} = \frac{x - x^3/6 + \dots}{x} = 1 - \frac{x^2}{6} + \dots \rightarrow 1$$

$$\boxed{\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1}$$

(b) $\lim_{x \rightarrow a} \frac{x^2 + (2-a)x - 2a}{x^2 - (a+1)x + a}$

Both numerator and denominator vanish at $x = a$ (verify: top gives $a^2 + 2a - a^2 - 2a = 0$; bottom gives $a^2 - a^2 - a + a = 0$).

Factor:

- Top: $x^2 + (2-a)x - 2a = (x-a)(x+2)$ [since roots are $x = a$ and $x = -2$]
- Bottom: $x^2 - (a+1)x + a = (x-a)(x-1)$ [since roots are $x = a$ and $x = 1$]

Therefore:

$$\lim_{x \rightarrow a} \frac{(x-a)(x+2)}{(x-a)(x-1)} = \lim_{x \rightarrow a} \frac{x+2}{x-1} = \frac{a+2}{a-1}$$

$$\boxed{\lim_{x \rightarrow a} \frac{x^2 + (2-a)x - 2a}{x^2 - (a+1)x + a} = \frac{a+2}{a-1}}$$

(Note: This assumes $a \neq 1$. If $a = 1$, both factors $(x-1)$ appear and further analysis is needed.)

(c) $\lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{(e^{ax} - 1)^2}$

Using Taylor expansions:

- $e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \dots$, so $e^{x^2} - 1 = x^2 + O(x^4)$
- $e^{ax} = 1 + ax + \frac{(ax)^2}{2} + \dots$, so $e^{ax} - 1 = ax + O(x^2)$
- $(e^{ax} - 1)^2 = a^2x^2 + O(x^3)$

Therefore:

$$\frac{e^{x^2} - 1}{(e^{ax} - 1)^2} = \frac{x^2 + O(x^4)}{a^2x^2 + O(x^3)} = \frac{1 + O(x^2)}{a^2 + O(x)} \rightarrow \frac{1}{a^2}$$

$$\boxed{\lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{(e^{ax} - 1)^2} = \frac{1}{a^2}}$$

(d) $\lim_{x \rightarrow \infty} x^n e^{-ax}$ for $a > 0$ and $n \in \mathbb{R}$

Write this as:

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^{ax}}$$

The exponential grows faster than any polynomial. To see this formally, expand:

$$e^{ax} = \sum_{k=0}^{\infty} \frac{(ax)^k}{k!}$$

For any fixed n , eventually the terms with $k > n$ dominate:

$$\frac{x^n}{e^{ax}} = \frac{x^n}{\sum_{k=0}^{\infty} \frac{a^k x^k}{k!}} < \frac{x^n}{\frac{a^k x^k}{k!}} = \frac{k!}{a^k} x^{n-k}$$

For $k > n$, this goes to 0 as $x \rightarrow \infty$.

Note: For positive integer n only, we could have also used L'Hôpital's rule n times:

$$\lim_{x \rightarrow \infty} \frac{x^n}{e^{ax}} = \lim_{x \rightarrow \infty} \frac{n!}{a^n e^{ax}} = 0$$

$$\boxed{\lim_{x \rightarrow \infty} x^n e^{-ax} = 0}$$