

Problem Set 9: Dirac Delta and Matrices

1. Evaluate the following integrals:

(a) $\int_0^2 x^2 \delta(x-1) dx$

(b) $\int_{-\infty}^y \delta(x) dx$ as a function of y

(c) $\int_0^5 \cos x \delta(x-\pi) dx$

(d) $\int_{-\infty}^{\infty} (3x-7)\delta(2x) dx$

(e) $\int_{-\infty}^{\infty} (3x-7)\delta'(x) dx$ where $\delta'(x) = \frac{d}{dx}\delta(x)$. (Hint: use integration by parts.)

(f) $\int_V (r^2 + \vec{r} \cdot \vec{a} + a^2) \delta^{(3)}(\vec{r} - \vec{a}) d^3r$ where the volume V is all space and $\vec{a}$ is a fixed vector.

(g) $\int_V |\vec{r} - \vec{b}|^2 \delta^{(3)}(5\vec{r}) d^3r$ where V is a cube of side length 2 centered on the origin and $\vec{b} = 3\hat{x} + 4\hat{y}$.

(h) The same integral as in the previous part but now with the volume V running over a cube of side length 10.

2. Evaluate

$$J = \int_V e^{-r} \vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) d\tau \quad (1)$$

where V is a ball of radius R around the origin in two ways.

(a) Using integration by parts to transform it into a surface integral.

(b) Using the 3D Delta function.

3. Consider the matrices:

$$A = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad (2)$$

Of the 9 possible matrix products ($AA, AB, AC, BA, BB, BC, CA, CB, CC$), determine which are well-defined and evaluate those that are.

4. Consider the matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \quad (3)$$

- (a) Evaluate A^2 .
 - (b) Evaluate AB .
 - (c) Evaluate BA .
 - (d) What is the matrix inverse of B ? I.e. what matrix C has the property that $CB = I$ for the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$?
 - (e) Let $M_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ for any number a . What is $M_a M_b$ for arbitrary numbers a and b ?
5. Alice has A dollars and Bob has B dollars. Together, they have \$100, but Alice has 4 times as much money as Bob.
- (a) Write a system of two linear equations for how much money Alice (A) and Bob (B) have.
 - (b) Rewrite your system as a matrix equation $Mx = c$ where $x = \begin{pmatrix} A \\ B \end{pmatrix}$ is a column vector of the unknowns. What are the 2×2 matrix M and column vector c ?

6. Let σ_i be the three Pauli matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4)$$

for $i = x, y, z$, and let p^i be the components of a 3D vector. Show that $(\sum_i \sigma_i p^i)^2 = (\vec{\sigma} \cdot \vec{p})^2 = \vec{p}^2 I$ where I is the 2×2 identity matrix.

7. Show that the most general 2×2 matrix whose square is zero can be written as

$$\begin{pmatrix} ab & b^2 \\ -a^2 & -ab \end{pmatrix} \quad (5)$$

where a and b are real or complex numbers. (Note: $A^2 = 0$ does not imply $A = 0$.)