

Problem Set 8: Fundamental Theorems, Potentials, and Curvilinear Coordinates

1. Prove the following integration by parts formula for the volume integral of $\vec{B} \cdot (\vec{\nabla} \times \vec{A})$:

$$\int_{\mathcal{V}} \vec{B} \cdot (\vec{\nabla} \times \vec{A}) dv = \int_{\mathcal{V}} \vec{A} \cdot (\vec{\nabla} \times \vec{B}) dv + \oint_{\mathcal{S}} (\vec{A} \times \vec{B}) \cdot d\vec{\sigma} \quad (1)$$

where $\mathcal{S}$ is the surface bounding the volume $\mathcal{V}$. (Hint: consider $\vec{\nabla} \cdot (\vec{A} \times \vec{B})$.)

2. Suppose a radial force field $\vec{F}$ is given by

$$\vec{F} = r^n \hat{r} \quad (2)$$

By working in Cartesian coordinates, find

- (a) $\vec{\nabla} \times \vec{F}$
 - (b) A scalar potential $\phi(x, y, z)$ such that $\vec{F} = -\vec{\nabla}\phi$.
 - (c) For what value of n does the scalar potential diverge at both the origin and infinity?
3. In the absence of charge, the electrostatic potential ϕ satisfies the Laplace equation,

$$\nabla^2 \phi = 0 \quad (3)$$

where the electric field $\vec{E} = -\vec{\nabla}\phi$. A parallel plate capacitor is composed of two sheets of metal which extend infinitely in the xy -plane, one at $z = 0$ and one at $z = d$, with empty space in between. The voltage V across the capacitor is the difference in potential between the two plates (which are equipotentials).

- (a) Show that $\phi = Vz/d$ satisfies the Laplace equation and the boundary condition that $\phi(z = 0) = 0$ and $\phi(z = d) = V$ on the two plates.
 - (b) Determine the electric field $\vec{E}$ between the plates.
 - (c) Make a sketch of the electric field lines and the equipotentials between the plates.
4. Consider the vector potential $\vec{A} = \frac{1}{2}B_0\rho\hat{e}_\phi$ in cylindrical coordinates, where B_0 is a constant.
- (a) Compute the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ using the cylindrical coordinate curl formula.
 - (b) Compute the magnetic flux $\Phi = \int_S \vec{B} \cdot d\vec{\sigma}$ through a disk of radius R centered at the origin in the xy -plane.
 - (c) Now use Stokes' theorem to compute Φ by evaluating the line integral $\oint_C \vec{A} \cdot d\vec{l}$ around the boundary circle. Verify that you get the same answer.

5. The u, v, z curvilinear coordinate system is defined by

$$xy = u, \quad x^2 - y^2 = v, \quad z = z \quad (4)$$

The coordinate system defined this way is orthogonal (that is, the coordinate surfaces are orthogonal).

- (a) Describe the nature of the three families of coordinate surfaces.
 - (b) Sketch the u, v coordinate surfaces in the xy plane ($z = 0$).
 - (c) Indicate the direction of the local basis vectors $\hat{e}_u, \hat{e}_v$ in each of the 4 quadrants of the xy plane.
 - (d) Is the u, v, z coordinate system right handed ($\hat{e}_u \times \hat{e}_v = +\hat{e}_z$) or left handed ($\hat{e}_u \times \hat{e}_v = -\hat{e}_z$)?
6. Show that, in cylindrical coordinates,

$$\frac{\partial \hat{e}_\rho}{\partial \phi} = \hat{e}_\phi \quad (5)$$

$$\frac{\partial \hat{e}_\phi}{\partial \phi} = -\hat{e}_\rho \quad (6)$$

and that all other first derivatives of the unit vectors vanish with respect to the cylindrical coordinates (ρ, ϕ, z) . You may want to express the local basis in terms of the Cartesian basis first.

7. A rigid body is rotating around a fixed axis with angular velocity $\vec{\omega} = \omega \hat{z}$. Using spherical polar coordinates, evaluate

(a) $\vec{v} = \vec{\omega} \times \vec{r}$

(b) $\vec{\nabla} \times \vec{v}$

8. A certain force field is given in spherical polar coordinates by

$$\vec{F} = \hat{e}_r \frac{2P \cos \theta}{r^3} + \hat{e}_\theta \frac{P \sin \theta}{r^3} \quad (7)$$

where P is a constant.

- (a) Sketch the force field in the x - z plane.
- (b) Evaluate $\vec{\nabla} \times \vec{F}$. Is $\vec{F}$ conservative?
- (c) Calculate $\oint \vec{F} \cdot d\vec{r}$ for a unit circle on the $\theta = \pi/2$ plane. Is this consistent with your answer to part (b)?
- (d) If $\vec{F}$ is conservative, find a potential ϕ such that $\vec{F} = -\vec{\nabla}\phi$. Otherwise, state it doesn't exist.