

Problem Set 7: Integration and Fundamental Theorems

1. Suppose a force field is given by $\vec{F} = x^3\hat{x} + 2yz\hat{y} + y^2\hat{z}$. Calculate the work done by the force, $\int \vec{F} \cdot d\vec{r}$, if a particle moves from the origin to the point $(1, 1, 1)$ along each of the following straight line paths

- (a) $(0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, 1, 0) \rightarrow (1, 1, 1)$.
- (b) $(0, 0, 0) \rightarrow (0, 0, 1) \rightarrow (0, 1, 1) \rightarrow (1, 1, 1)$.
- (c) $(0, 0, 0) \rightarrow (1, 1, 1)$ (along the diagonal straight line).

Does the work done depend on the path (i.e. is the force non-conservative)?

2. Evaluate the surface integral

$$\frac{1}{3} \oint \vec{r} \cdot d\vec{\sigma} \quad (1)$$

over the surface of the unit cube in the first octant (i.e. with corners at the origin and $\hat{x}, \hat{y}, \hat{z}$). (Hint: three of the faces contribute zero and the other three contribute the same amount.)

3. A cube with sides of length 2 cm is centered on the origin of a Cartesian coordinate system (i.e. it stretches from -1 to 1 in each of the x, y, z directions). It has mass density $\rho = (1 - x^2)(1 - y^2)(1 - z^2)$ in units of grams/cm³.

- (a) Sketch a contour plot of the density in the xy -plane (at $z = 0$).
- (b) Set up and evaluate an integral for the total mass M of the cube.

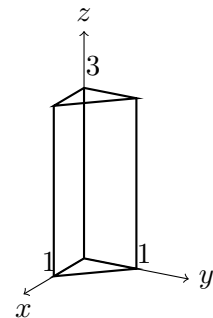
4. A long wire lies centered along the z -axis. Its cross section is square of side length a (so it stretches from $-a/2$ to $+a/2$ in the x and y directions). The wire carries a current density

$$\vec{J} = J_0 \left(\frac{x}{a} \right)^2 \hat{z} \quad (2)$$

for some constant J_0 .

- (a) Sketch the wire and the current density $\vec{J}$.
- (b) What integral represents the total current I carried by the wire? (I.e. the total amount of current crossing the xy -plane.)
- (c) Evaluate I .

5. A right triangular prism has geometry as shown to the right. The base is a right triangle with side lengths 1, and the prism has height 3. What is the flux $\oint \vec{v} \cdot d\vec{\sigma}$ of the field $\vec{v} = x\hat{x} + y\hat{y} + z\hat{z}$ out of the prism?



6. Show that

$$\frac{1}{3} \oint_{\Sigma} \vec{r} \cdot d\vec{\sigma} = V \quad (3)$$

where V is the volume enclosed by the surface Σ .

7. If $\vec{B} = \vec{\nabla} \times \vec{A}$, show that

$$\oint_{\Sigma} \vec{B} \cdot d\vec{\sigma} = 0 \quad (4)$$

for any closed surface Σ .