

Problem Set 6: Vector Derivatives and Vector Calculus

1. Find the gradients of the following scalar fields:
 - (a) $f(\vec{r}) = r^{-3/2}$
 - (b) $g(\vec{r}) = x^2 + y^3 + z^4$
 - (c) What is the magnitude of $\vec{\nabla} f$ (from part a) at the point $(2, 2, 1)$?
2. Suppose $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 = (x_1 - x_2)\hat{x} + (y_1 - y_2)\hat{y} + (z_1 - z_2)\hat{z}$ is the displacement vector from point 2 to point 1. Show that $\vec{\nabla}_1 r_{12}$, the gradient with respect to point 1 of the magnitude $r_{12} = |\vec{r}_{12}|$, is a unit vector pointing in the direction of $\vec{r}_{12}$.
3. Assuming the product rule for ordinary (partial) derivatives, show that $\vec{\nabla}(uv) = u\vec{\nabla}v + v\vec{\nabla}u$ where u, v are smooth scalar fields.
4. Assuming linearity of the ordinary (partial) derivative, show that the $\vec{\nabla}$ operator is linear. That is, show that $\vec{\nabla}(aF + bG) = a\vec{\nabla}F + b\vec{\nabla}G$ for arbitrary scalar fields F, G and constants a, b .
5. A rigid body rotates at constant angular velocity $\vec{\omega} = \omega\hat{z}$ about the z axis, where $\omega > 0$. The velocity of points on the rigid body at position $\vec{r}$ is given by $\vec{v} = \vec{\omega} \times \vec{r}$.
 - (a) Show that $\vec{v} = \omega(x\hat{y} - y\hat{x})$.
 - (b) Sketch $\vec{v}$ in the xy -plane using an arrow plot.
 - (c) Compute the divergence $\vec{\nabla} \cdot \vec{v}$.
 - (d) Compute the curl $\vec{\nabla} \times \vec{v}$.
6. Find the divergence and curl of $\vec{v} = r\hat{x}$.
7. Working in Cartesian coordinates, show that (hint: $[\vec{A} \times \vec{B}]_i = \epsilon_{ijk}A_jB_k$)
 - (a) the curl of a gradient is identically zero: $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$
 - (b) the divergence of a curl is identically zero: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$.
8. A particular scalar field ϕ satisfies Laplace's equation $\nabla^2\phi = 0$. Show that the gradient of this scalar field has neither curl nor divergence.
9. Show that $\vec{u} \times \vec{v}$ is solenoidal (divergence free) if $\vec{u}$ and $\vec{v}$ are both irrotational (curl free).
10. Suppose the curl of $\vec{A}$ equals the curl of $\vec{B}$.
 - (a) Show that $\vec{A}$ and $\vec{B}$ may differ by a constant.
 - (b) Show that $\vec{A}$ and $\vec{B}$ may differ by the gradient of a scalar field.

This (partially) generalizes the well-known result that two 1D functions with the same derivative may differ by a constant.