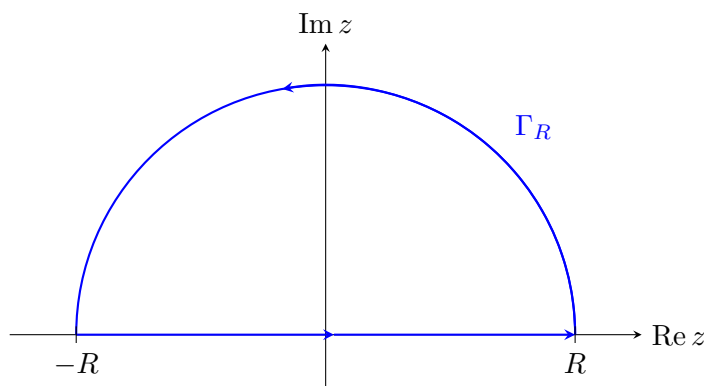


One More Complex Contour Integration

1. Complex contour integration can be quite powerful for evaluating difficult real integrals. Here is an example: Consider the function

$$f(z) = \frac{e^{iz}}{z^2 + 1}$$

and the closed contour C consisting of the real-axis segment $[-R, R]$ together with the semi-circular arc Γ_R of radius $R \gg 1$ in the upper half-plane, traversed counterclockwise (see figure).



- (a) Find the poles of $f(z)$. Which pole lies inside C ? Use the residue theorem to evaluate $\oint_C f(z) dz$.

- (b) Show that $\int_{\Gamma_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$.

Hint: On the upper semicircle, $z = Re^{i\theta}$ with $0 \leq \theta \leq \pi$. Show that $|e^{iz}| \leq 1$.

- (c) Use parts (a) and (b) together with Euler's formula to evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 1} dx$.

Vectors

2. Consider the following 3 vectors:

$$\vec{u} = \hat{e}_x + \hat{e}_y$$

$$\vec{v} = \hat{e}_y + \hat{e}_z$$

$$\vec{w} = \hat{e}_x + \hat{e}_z$$

- (a) On a 3D set of axes, sketch $\vec{u}$, $\vec{v}$, and $\vec{w}$ with all three vectors originating at the origin.
 - (b) What is the length of each vector?
 - (c) What is the angle between the vectors (in radians)?
 - (d) What is the volume of the parallelepiped whose edges are given by $\vec{u}$, $\vec{v}$, and $\vec{w}$?
3. The vector $\vec{A} \in \mathbb{R}^3$ whose magnitude is 1.732 units makes equal angles with the coordinate axes. Find A_x , A_y , and A_z .
4. Consider the following three vectors (written with respect to the standard basis)

$$\vec{e}'_1 \doteq \frac{1}{9} \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix} \quad \vec{e}'_2 \doteq \frac{1}{9} \begin{pmatrix} -7 \\ 4 \\ 4 \end{pmatrix} \quad \vec{e}'_3 \doteq \frac{1}{9} \begin{pmatrix} -4 \\ -8 \\ 1 \end{pmatrix} \quad (1)$$

- (a) Show that $\{\vec{e}'_i\}$ forms an orthonormal basis.
 - (b) Let $\vec{w} \doteq (1, 2, 3)^T$ in standard basis. Find the components w^i with respect to the $\vec{e}'_i$ basis.
5. Hubble found that distant galaxies are receding with a velocity proportional to their distance from where we are on the Earth. For the i 'th galaxy,

$$\vec{v}_i = H_0 \vec{r}_i \quad (2)$$

with us at the origin. Show that this recession of the galaxies from us does not imply that we are the center of the universe. Take the galaxy at $\vec{r}_1$ as a new origin and show that aliens living in this galaxy observe the same Hubble law.