

Problem Set 4: Complex Calculus

1. Determine where (in what regions of $\mathbb{C}$, if any) the following functions are analytic (i.e. where the Cauchy-Riemann conditions hold). Below, x and y refer to the real and imaginary parts of $z = x + iy$.

(a) $f = e^x \cos y + ie^y \cos x$

(b) $f = x^2 + y^2$

(c) $f = \frac{x^2 + y^2 + 3x - iy + 2}{x^2 + y^2 + 2x + 1}$

2. Locate and identify the type of any singularities in the following functions, if any. For each simple pole, determine its residue.

(a) $f = \frac{\cos z}{z}$

(b) $f = \frac{\sin z}{z}$

(c) $f = \sqrt{z - 2i}$

(d) $f = \frac{1}{z^2 + i}$

3. Show that the real $u(x, y)$ and imaginary $v(x, y)$ parts of an analytic function $f(x, y) = u(x, y) + iv(x, y)$ satisfy the two dimensional Laplace equation

$$\partial_x^2 u + \partial_y^2 u = 0 \tag{1}$$

and similarly for v . (Hint: consider the Cauchy-Riemann conditions.)

4. Consider the function $f(z) = z^3$. By explicit parametrization, compute $\int_C f(z) dz$ for the contours:

(a) C a straight line from 1 to i .

(b) C a quarter circle centered on the origin from 1 to i .

(c) Check these agree with the integral found by using the usual rules of antidifferentiation.

5. Using the Cauchy Residue Theorem, compute the following integrals where C is a closed circle of radius 2 about the origin:

(a) $\oint_C \frac{dz}{1 + z^2}$

(b) $\oint_C \frac{\sin z}{1 + z^2} dz$