

Problem Set 3: Complex Numbers

1. Write each of the following numbers in both Cartesian form $z = x + iy$ and polar form $z = re^{i\theta}$ by finding the real part $x = \operatorname{Re} z$, imaginary part $y = \operatorname{Im} z$, modulus $r = |z|$ and phase θ :

(a) $z_1 = \frac{2}{1+i}$

(b) $z_2 = (1 + \sqrt{3}i)^2$

(c) $z_3 = \frac{1 + \sqrt{3}i}{1 - \sqrt{3}i}$

(d) $z_4 = \frac{1-i}{1-\sqrt{3}i}$

(e) $z_5 = \cos \theta + i \sin \theta \quad \theta \in [0, 2\pi]$

(f) $z_6 = 3^i$

(g) $z_7 = \ln \frac{1+i}{1-i}$

2. Solve for x and y given

$$\frac{2+3i}{6+7i} + \frac{2}{x+iy} = 2+9i$$

3. Find the sum, product and quotient of z_1 and z_3 . You may find the result using whichever form (Cartesian or polar) is more convenient.
4. If you drive any two-terminal circuit built out of linear circuit elements (resistors, capacitors and inductors) with a (co)sinusoidally varying voltage given by (the real part of)

$$V(t) = V_0 e^{i\omega t}, \quad (1)$$

the resulting current oscillates at the same frequency

$$I(t) = I_0 e^{i\omega t}. \quad (2)$$

The complex impedance $Z(\omega)$ of the circuit is defined by (generalized Ohm's law)

$$V_0 = I_0 Z(\omega). \quad (3)$$

- (a) For the series RLC circuit we considered in class, show that $Z = R + i\omega L + 1/(i\omega C)$.
- (b) Suppose the real voltage drive is given by $\cos \omega t$ (i.e. with amplitude one volt) at angular frequency $\omega = 2\pi \times 60/\text{s}$. Sketch the real current response $I(t)$ if $R = 300 \Omega$, $L = 1 \text{ H}$ and $C = 1 \text{ F}$.
- (c) Sketch the real current response $I(t)$ as in part (b) if $\omega = 2\pi/\text{s}$.

5. Prove algebraically that, for $z, w \in \mathbb{C}$,
- (a) $\operatorname{Re} z \leq |z|$ and $\operatorname{Im} z \leq |z|$
 - (b) $|zw| = |z||w|$
 - (c) $|z + w|^2 = |z|^2 + |w|^2 + 2 \operatorname{Re}(zw^*)$
6. Given that $e^{iA}e^{iB} = e^{i(A+B)}$ derive the trigonometric formulas for $\cos(A+B)$ and $\sin(A+B)$ in terms of sines and cosines of A and B .