

1. Compute the following definite integrals. Be explicit about the rules you are using (e.g. for integration by parts indicate u , u' , v , v'). Make sure to check your answers by differentiation.

(a) $I(z) = \int_1^z dx (2x^3 - 2x + 3)$

(b) $I(z) = \int_0^z dx 2^x$

(c) $I(z) = \int_0^z dx \cosh(2x)$

(d) $I(z) = \int_0^z dx \frac{1}{3x+2}$

(e) $I(z) = \int_0^z dx x^2 \cos(2x)$

(f) $I(z) = \int_0^z dx x \cos(x^2 + \pi)$

(g) $I(z) = \int_0^z dx x^3 e^{-x^4}$

(h) $I(z) = \int_0^z dx \cosh^3 x$

(i) $I(z) = \int_0^z dx \frac{1}{\sqrt{a^2 - x^2}} \quad (|z| < a) \quad \text{Hint: } 1 - \sin^2 \theta = \cos^2 \theta$

2. Use integration by parts to find the indefinite integral of $\ln x = 1 \cdot \ln x$. Be explicit about your choice of u , u' , v , v' .
3. Consider the function,

$$F(n) = \int_0^\infty dx x^n e^{-x} \quad (1)$$

Using integration by parts, show that $F(n) = n F(n-1)$ and then use proof by induction to show that $F(n) = n!$ for $n \in \mathbb{N}$. What is $0!$ as defined by this integral?

4. Many important functions do not have elementary anti-derivatives but still have definite integrals (over say the whole real line) that are computable. One useful trick is to *differentiate with respect to a parameter*. Consider the general Gaussian integral (arguably the most important integral in theoretical physics),

$$I_n(a) = \int_{-\infty}^\infty dx x^n e^{-ax^2} \quad (2)$$

with $n \in \mathbb{N}$ and $a \in \mathbb{R}$. The integrals I_n converge for any parameter $a > 0$ but diverge badly otherwise. Let's see how far we can get evaluating them.

- (a) By considering $u = \sqrt{a}x$, show that $I_0(a) = \frac{C}{\sqrt{a}}$ for some dimensionless constant $C = I_0(1)$. It turns out that $C = \sqrt{\pi}$, which we will show later using 2D integration.
- (b) What is $I_n(a)$ for n odd?
- (c) Show by differentiating $I_0(a)$ with respect to a underneath the integral sign that

$$I_{2n}(a) = (-1)^n \frac{d^n I_0(a)}{da^n} \quad (3)$$

- (d) Evaluate the first few even $I_{2n}(a)$ using the formula in part (c) and then from the pattern guess the general formula for $I_{2n}(a)$.