

Problem Set 11: Eigensystems and Differential Equations

1. Determine the characteristic polynomial, eigenvalues and corresponding normalized eigenvectors of the matrices below. Make sure you choose orthogonal eigenvectors for any degenerate eigenvalues.
 - (a) The Pauli Matrices, $\sigma_x, \sigma_y, \sigma_z$.
 - (b) The 2×2 Identity Matrix, I .
 - (c) The 2×2 zero matrix.
 - (d) $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 7 \end{pmatrix}$
 - (e) $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -i \\ 0 & i & 1 \end{pmatrix}$
2. The matrix A has eigenvalues λ_i and corresponding eigenvectors v_i . Show that A^{-1} has the same eigenvectors but with eigenvalues $1/\lambda_i$.
3. A real square matrix A is called **orthogonal** if $A^{-1} = A^T$.
 - (a) Show that AB is orthogonal if A and B are orthogonal matrices.
 - (b) Show that A^{-1} is orthogonal if A is orthogonal.
 - (c) Show that $\det A = \pm 1$. The set of $n \times n$ orthogonal matrices is called the *orthogonal group* $O(n)$; the subset such that $\det A = +1$ is called the *special orthogonal group* $SO(n)$. (A group is a set which is closed under a multiplication operation and for which every element A has a multiplicative inverse.)
 - (d) Show from the definition that the most general element of $SO(2)$ can be written in the form

$$A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (1)$$
 for some angle θ . If A is the matrix of a linear map $\hat{A} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with respect to the standard basis, what is its geometric interpretation?
 - (e) Write an element of $O(2)$ such that $\det A = -1$. What is the geometric interpretation of your transformation?
4. Consider the operator $\hat{p} = -i \frac{d}{dx}$.
 - (a) Show that $\hat{p}$ is Hermitian when acting on the space of smooth functions on the interval $[0, 1]$ with periodic boundary conditions with inner product $\langle f|g \rangle = \int_0^1 dx f^*(x)g(x)$. The definition of hermiticity is that $\hat{H}$ satisfies $\langle f|\hat{H}g \rangle = \langle \hat{H}f|g \rangle$ for arbitrary functions f and g . (hint: integrate by parts).

- (b) Find the spectrum and corresponding normalized eigenfunctions of $\hat{p}$ by solving the eigenvalue equation:

$$\hat{p}f = \lambda f \quad (2)$$

Don't forget to impose the constraint that $f(0) = f(1)$!

The spectral theorem for Hermitian operators implies that this (infinite) set of eigenfunctions forms a complete orthonormal basis for smooth periodic functions on $[0, 1]$. That is, every smooth periodic function $g(x)$ can be represented in the Fourier basis you just derived.

5. A boat coasts to a stop due to a drag force proportional to v^n where v is the instantaneous velocity:

$$m \frac{dv}{dt} = -kv^n \quad (3)$$

- (a) For what values of n is this equation linear?
 - (b) For what values of n is it homogeneous?
 - (c) With $x(t = 0) = 0$ and $v(t = 0) = v_0$, find the velocity as a function of time. (Hint: separate variables.)
6. Find the general solution to $y''' - 2y'' + y' - 2y = 0$. (Hint: try an exponential ansatz $y \sim e^{cx}$.)
7. The concentration of a given radioactive isotope decays at a rate proportional to how much is left:

$$\frac{dN}{dt} = -\lambda N \quad (4)$$

where λ is a decay rate. In a radioactive series, one isotope decays into another which can then decay:

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \quad (5)$$

$$\frac{dN_2}{dt} = +\lambda_1 N_1 - \lambda_2 N_2 \quad (6)$$

At time $t = 0$, $N_1(0) = N_0$ and $N_2(0) = 0$.

- (a) Solve the first equation for $N_1(t)$.
- (b) Plug your solution to part (a) into the second equation and then solve for $N_2(t)$.