

Problem Set 10: Determinants and More Matrices

1. Evaluate the following determinants:

$$(a) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \quad (b) \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \quad (c) \begin{vmatrix} 0 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 3 & 0 \\ 0 & 3 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \end{vmatrix}$$

2. On a 3D set of axes, sketch the following 3 vectors: $\vec{u} = \hat{e}_x + \hat{e}_y$, $\vec{v} = \hat{e}_y + \hat{e}_z$, $\vec{w} = \hat{e}_x + \hat{e}_z$.

- (a) What is the length of each vector?
- (b) What is the angle between the vectors (in radians)?
- (c) What is the volume of the parallelepiped whose edges are given by $\vec{u}$, $\vec{v}$, and $\vec{w}$? (Hint: the determinant can tell you a volume).

3. The LU decomposition of a matrix A consists of writing $A = LU$ as the product of a lower L and upper U triangular matrix. Suppose that

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 7 & 0 & 0 \\ -2 & 12 & 3 & 0 \\ 5 & -5 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 & 10 \\ 0 & -1 & 3 & -5 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (1)$$

Compute $\det A$.

4. In quantum mechanics, angular momentum is represented by three matrices L_x , L_y , and L_z satisfying

$$[L_j, L_k] = i\epsilon_{jkl}L_l \quad (2)$$

Show that the trace of each angular momentum matrix vanishes.

- 5. If A and B are Hermitian matrices, show that $A + B$, $AB + BA$, and $i[A, B]$ are Hermitian matrices as well.
- 6. Prove that the following two definitions of linear independence of a set of vectors $\{\vec{v}_i\}$ are equivalent:
 - (a) If $\sum_i a_i \vec{v}_i = \vec{0}$, then all $a_i = 0$.
 - (b) No vector in $\{\vec{v}_i\}$ can be written as a linear combination of the others.
- 7. Let B be an $n \times n$ matrix. Show that if the columns of B are linearly dependent, then $\det(B) = 0$. (The converse statement is also true; you don't need to prove this. Thus the columns of a square matrix B are linearly independent if and only if $\det(B) \neq 0$.)

8. The *span* of a collection of vectors $S = \{\vec{v}_i\}$ is the set of vectors which can be made from linear combinations of the form $\vec{w} = a^i \vec{v}_i$ with coefficients a_i . A set S which is both linearly independent and spans is a (not necessarily orthonormal) basis for the span. Sketch each of the sets of vectors below (written in the standard basis) and indicate if the set is linearly independent, if it spans $\mathbb{R}^2$, and if it forms a basis for $\mathbb{R}^2$.

(a) $\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \vec{w} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$

(b) $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \vec{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

(c) $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

9. Let P_n be the set of polynomials in x with complex coefficients of degree $\leq n$. Recall: the degree of a polynomial $p(x)$ is the highest power of x appearing in $p(x)$.

(a) Show that P_n forms a vector space over $\mathbb{C}$.

(b) Argue that the set of monomials $S = \{1, x, x^2, \dots, x^n\}$ forms a basis for P_n .

(c) What is the dimension of P_n ?

(d) The derivative operator $\frac{d}{dx} : P_n \rightarrow P_n$ is a linear map on the space of polynomials. Find the matrix D representing $\frac{d}{dx}$ with respect to the monomial basis on $P_3[x]$.

(e) What is $\det D$?

(f) Is $\frac{d}{dx}$ invertible on P_n ?

10. Given a set of linearly independent vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, it is always possible to orthogonalize them into a new set of orthonormal vectors $\hat{e}_1 \dots \hat{e}_n$ which span the same space. A classic recipe for producing such a basis from the $\vec{v}_i$ is given by the *Gram-Schmidt Procedure*, described in Shankar on page 244 (see also Discussion 11). Use Gram-Schmidt to find an orthonormal basis from:

(a) $\vec{v}_1 = 3\hat{x} + 4\hat{y}, \vec{v}_2 = 2\hat{x} - 5\hat{y}$ in the plane.

(b)

$$|v_1\rangle = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \quad |v_2\rangle = \begin{pmatrix} 0 \\ 2 \\ i \end{pmatrix} \quad |v_3\rangle = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

in a 3 dimensional complex Hilbert space (the column vectors are in the standard basis).

- (c) The space spanned by the polynomials $\{p_0(x) = 1, p_1(x) = x, p_2(x) = x^2\}$ with inner product given by $\langle p|q \rangle = \int_{-1}^1 dx p^*(x)q(x)$.

A note on inner products of functions: The inner product $\langle p|q \rangle$ plays the same role for functions that the dot product plays for vectors. Two functions are orthogonal if $\langle p|q \rangle = 0$, and a function's norm (its "length") is $\|p\| = \sqrt{\langle p|p \rangle}$. An orthonormal basis of functions $\{e_i(x)\}$ therefore satisfies $\langle e_i|e_j \rangle = \delta_{ij}$, just like an orthonormal basis of vectors.