

PY 355: Methods of Theoretical Physics
Spring 2024 (Practice)

Name (Print): _____

Final

ID: _____

Time Limit: 120 minutes

- Do not turn this page until told to do so.
- Hand your exam in before leaving the room.
- Please write your name and student ID on the first page.
- This is a closed book exam. **No books, notes or calculators allowed.**
- Write all your work on these sheets. If you need more space, you may use the back of the front page; clearly indicate when you have done this.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided.
- **Mysterious or unsupported answers will not receive full credit.**

Question	Points	Score
1	5	
2	5	
3	10	
4	10	
5	35	
6	20	
7	30	
8	10	
9	35	
Total:	160	

1. The core of the earth is denser than the mantle (outer layer). As a very rough model, the mass density can be taken to be spherically symmetric and quadratically decreasing with radial distance from the center:

$$\rho(r) = \rho_0 \left(1 - \frac{1}{2} \left(\frac{r}{R} \right)^2 \right) \quad (1)$$

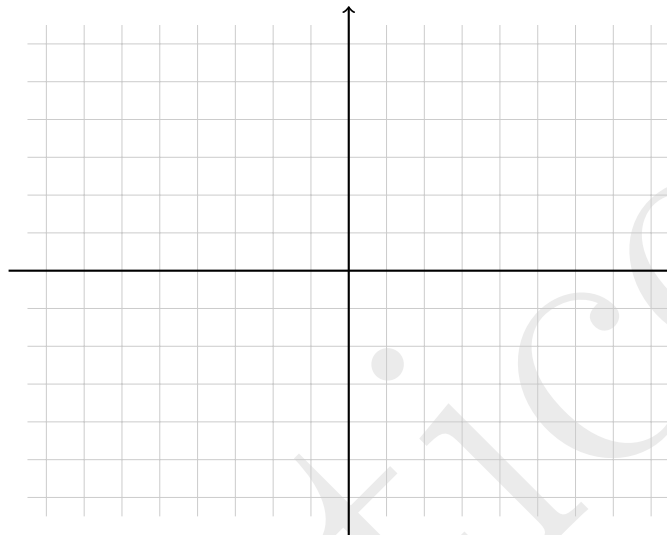
where R is the radius of the earth and ρ_0 is the density at the center. Within this model, what is the mass of the earth in terms of ρ_0 and R ?

2. Which, if any, of the following functions are analytic as a function of $z = x + iy$?

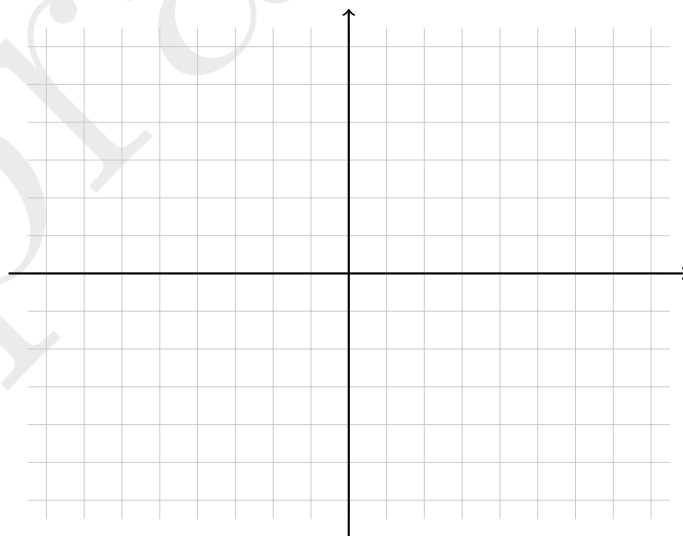
- A. $f_1(x, y) = x^2 + y^2 + 2ixy$
- B. $f_2(x, y) = x^2 - y^2 + 2ixy$
- C. $f_3(x, y) = x^2 + y^2 - 2ixy$
- D. $f_4(x, y) = x^2 - y^2 - 2ixy$

3. In the complex z plane, mark any zeros or poles and sketch the contours of constant phase (constant color in the Jupyter homework) of the function

$$f(z) = \frac{1}{z - i}$$



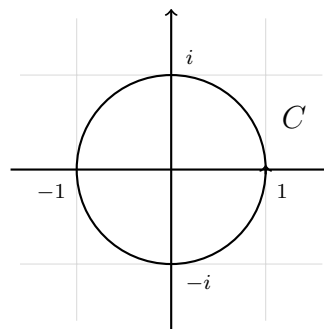
4. Sketch $f(x) = e^{x/(1-x)}$ as a function of $x \in \mathbb{R}$. Make sure to label any important features.



5. Evaluate the following expressions. Justify your answers.

- (a) For C the unit circle oriented counterclockwise,

$$\oint_C z \, dz$$



- (b) For the same contour C as in (a),

$$\oint_C \bar{z} \, dz$$

- (c) For the same contour C as in (a),

$$\oint_C \frac{\cos z}{z} \, dz$$

(d) $\int_{-2}^2 (x^2 + 2x + 1)(\delta(x - 1) + \delta(x - 3)) dx$

(e) $\int_{-\infty}^{\infty} x^3 \delta'(x - 1) dx$

(f) $\int_{-\infty}^{\infty} \frac{e^{-x}}{x} \delta(x^2 - 9) dx$

(g) $\int_{B(R)} e^{-\lambda r} \vec{\nabla} \cdot \left(\frac{\hat{\mathbf{r}}}{r} \right) d\tau$ where $B(R)$ is the ball of radius R .

6. Find the eigenvalues and a complete set of orthonormal eigenvectors for the matrix:

$$H = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \quad (2)$$

practice

7. Consider the space H of functions on the interval $(-\pi, \pi)$ spanned by $f_k(x) = e^{ikx}$ for $k = -3, -2, -1, 0, 1, 2, 3$.

(a) What dimension is H ?

(b) Expand the function $1 + \cos(2x)$ in the f_k basis.

(c) Suppose the function $g(x) \in H$ is real valued. What condition does this impose on the coefficients g^k in the expansion $g(x) = \sum_k g^k f_k(x)$?

- (d) Write the matrix representing the derivative map $D = \frac{d}{dx}$ acting on H .
- (e) Is D invertible? If yes, invert it. If not, explain why not.
- (f) Suppose we define an inner product on H by $\langle f|g \rangle = \int_{-\pi}^{\pi} dx f^*(x)g(x)$. Is the basis f_k orthonormal?

8. In a centrifuge, a fluid is undergoing uniform circular motion with angular frequency ω in a bucket of radius R . Neglecting gravity, the pressure P inside the fluid satisfies the force balance equation

$$-\vec{\nabla}P = -\rho\omega^2r\hat{r} \quad (3)$$

where ρ is the mass density (and r and $\hat{r}$ are the radius in *cylindrical* coordinates!). By the cylindrical symmetry, the pressure and density are only functions of r , so the equation reduces to:

$$\frac{dP}{dr} = \rho\omega^2r \quad (4)$$

- (a) For an incompressible fluid, $\rho(r) = \rho_0$ is a constant. Solve for the pressure profile $P(r)$ assuming $P(r = 0) = P_0$.

- (b) Real fluids are compressible: the density increases with pressure. In the simplest case, the ideal gas law implies that they are linearly related,

$$\rho = \alpha P \quad (5)$$

where α is a constant. This leads to the equation,

$$\frac{dP}{dr} = \alpha \omega^2 r P \quad (6)$$

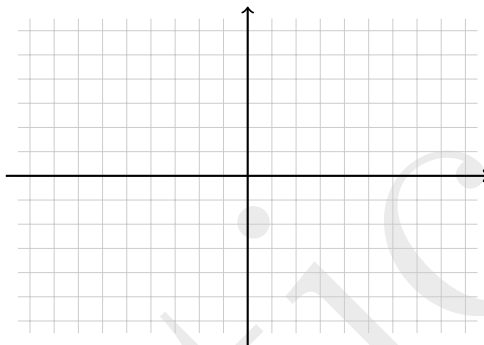
Solve this equation for $P(r)$, again assuming $P(r = 0) = P_0$.

9. A cylindrical bottle of radius R contains superfluid helium up to a height $z = L$ (the bottom of the bottle is at $z = 0$). The helium is stirred to produce a vortex in which the flow circulates about the cylindrical axis (the z -axis) with velocity

$$\vec{v} = v_0 \frac{\hat{\phi}}{s} \quad (7)$$

in cylindrical coordinates. The mass density of the helium is ρ_0 .

- (a) Sketch $\vec{v}$ in the x - y plane.



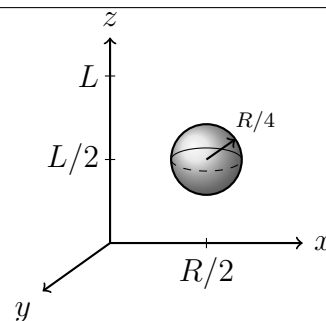
- (b) What is $\vec{\nabla} \cdot \vec{v}$?

(c) What is the total circulation $\oint_{S^1(R)} \vec{v} \cdot d\vec{r}$ around the circle of radius R in the XY plane?

(d) What is $\vec{\nabla} \times \vec{v}$?

(e) Can $\vec{v}$ be written as the gradient of a scalar field f ? If so, find f . If not, explain why not.

- (f) What is the total flux of the fluid per unit time out of the sphere of radius $R/4$ centered on the point $(s, \phi, z) = (R/2, 0, L/2)$?



- (g) The kinetic energy of the flow is given by adding up the kinetic energy from each chunk of mass $dm = \rho_0 d^3x$:

$$E = \frac{1}{2} \int dm \vec{v}^2 \quad (8)$$

Evaluate the kinetic energy. (Hint: since the velocity diverges as $s \rightarrow 0$, you may need to ‘cut out’ a small cylinder of radius a from the domain of integration – physically this simply acknowledges that velocities never actually diverge and something different must be happening at short distances near the vortex core.)

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\tau = dx dy dz$

$$\text{Gradient :} \quad \nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence :} \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl :} \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$$

$$\text{Laplacian :} \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin \theta dr d\theta d\phi$

$$\text{Gradient :} \quad \nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\text{Divergence :} \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned} \text{Curl :} \quad \nabla \times \mathbf{v} = & \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} \\ & + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \end{aligned}$$

$$\text{Laplacian :} \quad \nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$

$$\text{Gradient :} \quad \nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence :} \quad \nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl :} \quad \nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$$

$$\text{Laplacian :} \quad \nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$