

PY 355: Methods of Theoretical Physics
Spring 2024 (Practice)

Name (Print): _____

Final — Solutions

ID: _____

Time Limit: 120 minutes

- Do not turn this page until told to do so.
- Hand your exam in before leaving the room.
- Please write your name and student ID on the first page.
- This is a closed book exam. **No books, notes or calculators allowed.**
- Write all your work on these sheets. If you need more space, you may use the back of the front page; clearly indicate when you have done this.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided.
- **Mysterious or unsupported answers will not receive full credit.**

Question	Points	Score
1	5	
2	5	
3	10	
4	10	
5	35	
6	20	
7	30	
8	10	
9	35	
Total:	160	

1. The core of the earth is denser than the mantle (outer layer). As a very rough model, the mass density can be taken to be spherically symmetric and quadratically decreasing with radial distance from the center:

$$\rho(r) = \rho_0 \left(1 - \frac{1}{2} \left(\frac{r}{R} \right)^2 \right) \quad (1)$$

where R is the radius of the earth and ρ_0 is the density at the center. Within this model, what is the mass of the earth in terms of ρ_0 and R ?

Solution: Spherical volume integral:

$$\begin{aligned} M &= \int d^3x \rho = \int_0^R r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \rho_0 \left(1 - \frac{1}{2} (r/R)^2 \right) = 4\pi \rho_0 \int_0^R \left(r^2 - \frac{1}{2} \frac{r^4}{R^2} \right) dr \\ &= 4\pi \rho_0 \left(\frac{R^3}{3} - \frac{1}{2R^2} \cdot \frac{R^5}{5} \right) = 4\pi \rho_0 R^3 \left(\frac{1}{3} - \frac{1}{10} \right) = \boxed{\frac{14}{15} \pi \rho_0 R^3}. \end{aligned}$$

2. Which, if any, of the following functions are analytic as a function of $z = x + iy$?

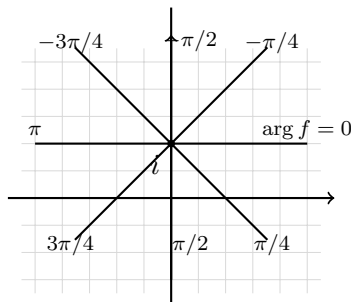
- A. $f_1(x, y) = x^2 + y^2 + 2ixy$
- B. $f_2(x, y) = x^2 - y^2 + 2ixy$
- C. $f_3(x, y) = x^2 + y^2 - 2ixy$
- D. $f_4(x, y) = x^2 - y^2 - 2ixy$

Solution: Cauchy–Riemann conditions: $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$. For all four functions $\partial_y v = 2x$ for A, B and $-2x$ for C, D, while $\partial_x u = 2x$ universally, ruling out C and D. Then $\partial_y u = 2y$ for A, $-2y$ for B; $-\partial_x v = -2y$ for A, B. So only f_2 satisfies both equations. **Only B is analytic** (in fact $f_2 = z^2$).

3. In the complex z plane, mark any zeros or poles and sketch the contours of constant phase (constant color in the Jupyter homework) of the function

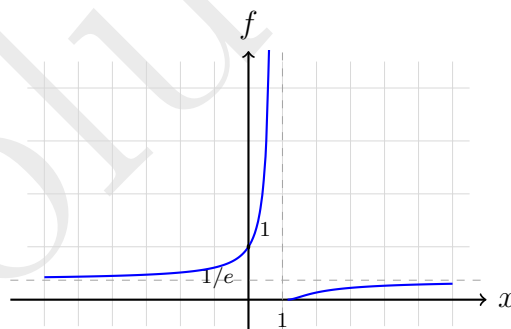
$$f(z) = \frac{1}{z - i}$$

Solution: Write $z - i = |z - i| e^{i\theta}$, so $f(z) = |z - i|^{-1} e^{-i\theta}$, i.e., $\arg f = -\arg(z - i)$. Constant-phase contours are *rays* emanating from the pole at $z = i$.



4. Sketch $f(x) = e^{x/(1-x)}$ as a function of $x \in \mathbb{R}$. Make sure to label any important features.

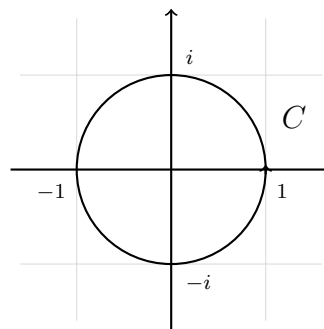
Solution: $f(0) = 1$. As $|x| \rightarrow \infty$, $x/(1-x) \rightarrow -1$ so $f \rightarrow e^{-1}$ (horizontal asymptote both sides). Essential singularity at $x = 1$: $x \rightarrow 1^-$ gives $f \rightarrow +\infty$; $x \rightarrow 1^+$ gives $f \rightarrow 0^+$.



5. Evaluate the following expressions. Justify your answers.

- (a) For C the unit circle oriented counterclockwise,

$$\oint_C z \, dz$$



Solution: z is entire (analytic everywhere), so by the Cauchy–Goursat theorem $\oint_C z \, dz = 0$.

- (b) For the same contour C as in (a),

$$\oint_C \bar{z} \, dz$$

Solution: Parametrize $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$, $\bar{z} = e^{-i\theta}$:

$$\oint_C \bar{z} \, dz = \int_0^{2\pi} e^{-i\theta} i e^{i\theta} d\theta = i \int_0^{2\pi} d\theta = 2\pi i.$$

($\bar{z}$ is not analytic, so Cauchy doesn't apply.)

- (c) For the same contour C as in (a),

$$\oint_C \frac{\cos z}{z} \, dz$$

Solution: Simple pole at $z = 0$ with residue $\cos 0 = 1$. By the residue theorem, $\oint_C \frac{\cos z}{z} \, dz = 2\pi i \cdot 1 = 2\pi i$.

(d) $\int_{-2}^2 (x^2 + 2x + 1)(\delta(x - 1) + \delta(x - 3)) dx$

Solution: Only $x = 1$ lies inside the integration domain $[-2, 2]$; $x = 3$ is outside, so its δ contributes nothing. Therefore $(1)^2 + 2(1) + 1 = 4$.

(e) $\int_{-\infty}^{\infty} x^3 \delta'(x - 1) dx$

Solution: Integration by parts: $\int x^3 \delta'(x - 1) dx = - \int 3x^2 \delta(x - 1) dx = -3(1)^2 = -3$.

(f) $\int_{-\infty}^{\infty} \frac{e^{-x}}{x} \delta(x^2 - 9) dx$

Solution: Use $\delta(g(x)) = \sum_i \delta(x - x_i)/|g'(x_i)|$ with $g(x) = x^2 - 9$, $g'(\pm 3) = \pm 6$: $\delta(x^2 - 9) = \frac{1}{6}[\delta(x - 3) + \delta(x + 3)]$. Thus

$$\int \frac{e^{-x}}{x} \delta(x^2 - 9) dx = \frac{1}{6} \left[\frac{e^{-3}}{3} + \frac{e^3}{-3} \right] = \frac{1}{18} (e^{-3} - e^3) = -\frac{1}{9} \sinh 3.$$

(g) $\int_{B(R)} e^{-\lambda r} \vec{\nabla} \cdot \left(\frac{\hat{\mathbf{r}}}{r} \right) d\tau$ where $B(R)$ is the ball of radius R .

Solution: In spherical coordinates, $\vec{\nabla} \cdot (\hat{\mathbf{r}}/r) = \frac{1}{r^2} \partial_r (r^2 \cdot \frac{1}{r}) = \frac{1}{r^2}$. So

$$\int_{B(R)} e^{-\lambda r} \frac{1}{r^2} r^2 \sin \theta dr d\theta d\phi = 4\pi \int_0^R e^{-\lambda r} dr = \frac{4\pi}{\lambda} (1 - e^{-\lambda R}).$$

6. Find the eigenvalues and a complete set of orthonormal eigenvectors for the matrix:

$$H = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \quad (2)$$

Solution: Characteristic polynomial.

$$\det(H - \lambda I) = (1 - \lambda)^3 - 2 - 3(1 - \lambda) = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda + 1)(\lambda - 2)^2.$$

Eigenvalues: $\lambda = -1, 2, 2$ (the eigenvalue 2 is doubly degenerate).

Eigenvectors. For $\lambda = -1$, by inspection $H(1, 1, 1)^T = (-1, -1, -1)^T = -(1, 1, 1)^T$, so

$$\mathbf{v}_{-1} = \frac{1}{\sqrt{3}}(1, 1, 1)^T.$$

For the degenerate $\lambda = 2$ subspace we need an orthonormal pair within the plane orthogonal to $\mathbf{v}_{-1}$ (H is symmetric so this plane is automatically the $\lambda = 2$ eigenspace); e.g.

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}(1, -1, 0)^T, \quad \mathbf{v}'_2 = \frac{1}{\sqrt{6}}(1, 1, -2)^T.$$

Eigenvalue check: $H(1, -1, 0)^T = (2, -2, 0)^T = 2(1, -1, 0)^T$ and $H(1, 1, -2)^T = (2, 2, -4)^T = 2(1, 1, -2)^T$.

Orthogonality check: $\mathbf{v}_{-1} \cdot \mathbf{v}_2 \propto 1 - 1 + 0 = 0$, $\mathbf{v}_{-1} \cdot \mathbf{v}'_2 \propto 1 + 1 - 2 = 0$, $\mathbf{v}_2 \cdot \mathbf{v}'_2 \propto 1 - 1 + 0 = 0$.

7. Consider the space H of functions on the interval $(-\pi, \pi)$ spanned by $f_k(x) = e^{ikx}$ for $k = -3, -2, -1, 0, 1, 2, 3$.

(a) What dimension is H ?

Solution: Seven basis vectors ($k = -3, \dots, 3$), so $\dim H = 7$.

(b) Expand the function $1 + \cos(2x)$ in the f_k basis.

Solution: $1 + \cos(2x) = 1 + \frac{1}{2}(e^{2ix} + e^{-2ix}) = f_0 + \frac{1}{2}f_2 + \frac{1}{2}f_{-2}$.

(c) Suppose the function $g(x) \in H$ is real valued. What condition does this impose on the coefficients g^k in the expansion $g(x) = \sum_k g^k f_k(x)$?

Solution: g real $\Leftrightarrow g(x) = g(x)^*$, i.e. $\sum_k g^k e^{ikx} = \sum_k (g^k)^* e^{-ikx} = \sum_{k'} (g^{-k'})^* e^{ik'x}$.

Matching coefficients: $\boxed{g^k = (g^{-k})^*}$.

- (d) Write the matrix representing the derivative map $D = \frac{d}{dx}$ acting on H .

Solution: Since $Df_k = ik f_k$, the basis $\{f_k\}$ diagonalizes D . Ordering the basis as $(f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, f_3)$,

$$D = \text{diag}(-3i, -2i, -i, 0, i, 2i, 3i).$$

- (e) Is D invertible? If yes, invert it. If not, explain why not.

Solution: No. $\det D = 0$ (the $k = 0$ diagonal entry vanishes); equivalently the constant function f_0 is a non-trivial element of $\ker D$.

- (f) Suppose we define an inner product on H by $\langle f|g \rangle = \int_{-\pi}^{\pi} dx f^*(x)g(x)$. Is the basis f_k orthonormal?

Solution: $\langle f_k|f_{k'} \rangle = \int_{-\pi}^{\pi} e^{i(k'-k)x} dx = 2\pi \delta_{kk'}$. **Orthogonal but not normalized;** the unit-norm basis would be $f_k/\sqrt{2\pi}$.

8. In a centrifuge, a fluid is undergoing uniform circular motion with angular frequency ω in a bucket of radius R . Neglecting gravity, the pressure P inside the fluid satisfies the force balance equation

$$-\vec{\nabla}P = -\rho\omega^2r\hat{r} \quad (3)$$

where ρ is the mass density (and r and $\hat{r}$ are the radius in *cylindrical* coordinates!). By the cylindrical symmetry, the pressure and density are only functions of r , so the equation reduces to:

$$\frac{dP}{dr} = \rho\omega^2r \quad (4)$$

- (a) For an incompressible fluid, $\rho(r) = \rho_0$ is a constant. Solve for the pressure profile $P(r)$ assuming $P(r=0) = P_0$.

Solution: Direct integration: $dP = \rho_0\omega^2r dr \Rightarrow P(r) = \frac{1}{2}\rho_0\omega^2r^2 + C$. The boundary condition $P(0) = P_0$ fixes $C = P_0$, so

$$P(r) = P_0 + \frac{1}{2}\rho_0\omega^2r^2.$$

- (b) Real fluids are compressible: the density increases with pressure. In the simplest case, the ideal gas law implies that they are linearly related,

$$\rho = \alpha P \quad (5)$$

where α is a constant. This leads to the equation,

$$\frac{dP}{dr} = \alpha \omega^2 r P \quad (6)$$

Solve this equation for $P(r)$, again assuming $P(r = 0) = P_0$.

Solution: Separable: $\frac{dP}{P} = \alpha \omega^2 r dr \Rightarrow \ln P = \frac{1}{2} \alpha \omega^2 r^2 + \tilde{C}$, so $P(r) = \tilde{C}' e^{\alpha \omega^2 r^2 / 2}$.
Imposing $P(0) = P_0$ gives $\tilde{C}' = P_0$:

$$P(r) = P_0 \exp\left(\frac{1}{2} \alpha \omega^2 r^2\right).$$

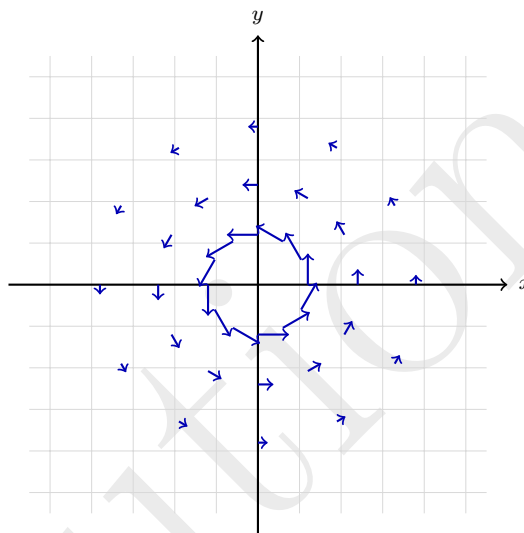
9. A cylindrical bottle of radius R contains superfluid helium up to a height $z = L$ (the bottom of the bottle is at $z = 0$). The helium is stirred to produce a vortex in which the flow circulates about the cylindrical axis (the z -axis) with velocity

$$\vec{v} = v_0 \frac{\hat{\phi}}{s} \quad (7)$$

in cylindrical coordinates. The mass density of the helium is ρ_0 .

- (a) Sketch $\vec{v}$ in the x - y plane.

Solution: Tangential field circulating counterclockwise about the z -axis, with magnitude $\propto 1/s$ (longer arrows near the axis, shorter far away).



- (b) What is $\vec{\nabla} \cdot \vec{v}$?

Solution: In cylindrical coordinates, $\vec{\nabla} \cdot \vec{v} = \frac{1}{s} \partial_s (s v_s) + \frac{1}{s} \partial_\phi v_\phi + \partial_z v_z = 0 + \frac{1}{s} \partial_\phi (v_0/s) + 0 = 0$.

- (c) What is the total circulation $\oint_{S^1(R)} \vec{v} \cdot d\vec{r}$ around the circle of radius R in the XY plane?

Solution: On the circle $z = 0$, $s = R$, $\phi \in [0, 2\pi]$ and $d\vec{r} = R d\phi \hat{\phi}$:

$$\oint \vec{v} \cdot d\vec{r} = v_0 \int_0^{2\pi} \frac{\hat{\phi}}{R} \cdot R \hat{\phi} d\phi = 2\pi v_0.$$

- (d) What is $\vec{\nabla} \times \vec{v}$?

Solution: For $s > 0$ the curl vanishes by direct calculation: $(\vec{\nabla} \times \vec{v})_z = \frac{1}{s} \partial_s (s v_\phi) = \frac{1}{s} \partial_s (v_0) = 0$. But this can't be the whole story: by Stokes' theorem the curl integrated over any disk in the xy -plane centered on the axis must reproduce the circulation $2\pi v_0$ from part (c), independent of R . The missing piece is a singular contribution localized on the vortex line $s = 0$:

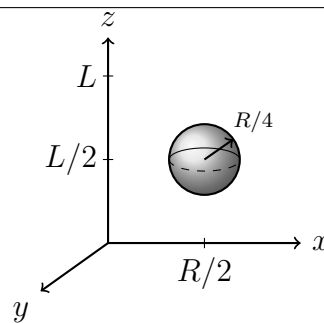
$$\vec{\nabla} \times \vec{v} = 2\pi v_0 \delta^{(2)}(x, y) \hat{z},$$

where $\delta^{(2)}(x, y)$ is the two-dimensional delta function in the xy -plane.

- (e) Can $\vec{v}$ be written as the gradient of a scalar field f ? If so, find f . If not, explain why not.

Solution: No — a gradient field has zero circulation around any closed loop, but $\vec{v}$ has circulation $2\pi v_0 \neq 0$ around any loop encircling the axis.

- (f) What is the total flux of the fluid per unit time out of the sphere of radius $R/4$ centered on the point $(s, \phi, z) = (R/2, 0, L/2)$?



Solution: Zero. The sphere does not enclose the singular axis, and away from the axis $\vec{\nabla} \cdot \vec{v} = 0$. By the divergence theorem the flux vanishes.

- (g) The kinetic energy of the flow is given by adding up the kinetic energy from each chunk of mass $dm = \rho_0 d^3x$:

$$E = \frac{1}{2} \int dm \vec{v}^2 \quad (8)$$

Evaluate the kinetic energy. (Hint: since the velocity diverges as $s \rightarrow 0$, you may need to ‘cut out’ a small cylinder of radius a from the domain of integration.)

Solution: In cylindrical coordinates with $\vec{v}^2 = v_0^2/s^2$ and $d^3x = s ds d\phi dz$:

$$E = \frac{1}{2} \rho_0 \int_0^L dz \int_0^{2\pi} d\phi \int_a^R \frac{v_0^2}{s^2} s ds = \frac{1}{2} \rho_0 v_0^2 \cdot L \cdot 2\pi \cdot \int_a^R \frac{ds}{s} = \pi \rho_0 v_0^2 L \ln(R/a).$$

The kinetic energy diverges logarithmically as $a \rightarrow 0$ — physically a finite vortex-core size a regularizes it.

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}; \quad d\tau = dx dy dz$

$$\text{Gradient :} \quad \nabla t = \frac{\partial t}{\partial x} \hat{\mathbf{x}} + \frac{\partial t}{\partial y} \hat{\mathbf{y}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence :} \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl :} \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{z}}$$

$$\text{Laplacian :} \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}}; \quad d\tau = r^2 \sin \theta dr d\theta d\phi$

$$\text{Gradient :} \quad \nabla t = \frac{\partial t}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}}$$

$$\text{Divergence :} \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$$

$$\begin{aligned} \text{Curl :} \quad \nabla \times \mathbf{v} = & \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{\mathbf{r}} \\ & + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\boldsymbol{\theta}} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \end{aligned}$$

$$\text{Laplacian :} \quad \nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds \hat{\mathbf{s}} + s d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}}; \quad d\tau = s ds d\phi dz$

$$\text{Gradient :} \quad \nabla t = \frac{\partial t}{\partial s} \hat{\mathbf{s}} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z} \hat{\mathbf{z}}$$

$$\text{Divergence :} \quad \nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\text{Curl :} \quad \nabla \times \mathbf{v} = \left[\frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\boldsymbol{\phi}} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{\mathbf{z}}$$

$$\text{Laplacian :} \quad \nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$