

1 Integrals with the Dirac Delta

Compute the following integrals containing the Dirac delta,

(a)

$$\int_0^{2\pi} \cos(\pi x) \delta(x - 7) dx ;$$

(b)

$$\int_0^{2\pi} x^2 \delta(x - 1) dx ;$$

(c)

$$\int_{-\infty}^a \delta(x) dx .$$

2 Delta function with complicated arguments

- (a) When the argument of the delta function is not just $(x - x_0)$, do a change of variable until it is. Use this method to show that

$$\delta(kx - x_0) = \frac{1}{|k|} \delta(x - x_0/k)$$

- (b) More generally, when a function $g(x)$ has simple zeros at x_i , that is, $g(x_i) = 0$ but $g'(x_i) \neq 0$, then

$$\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x - x_i).$$

- i. Express $\delta(x^2 - a^2)$ in terms of simple δ functions.

- ii. Compute $\int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} \delta(\sin(\pi x)) dx$. As a bonus challenge, evaluate the resulting series.

3 Dirac delta and integration by parts

Evaluate the following integral

$$I = \iiint_V (r^2 + R^2) \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) d\tau$$

where V is a ball (i.e. the volume enclosed by a spherical surface) of radius R centered at the origin.

(a) Use the definition of the 3D Dirac delta function

$$\delta^3(\mathbf{r}) = \frac{1}{4\pi} \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right)$$

to evaluate the integral directly.

(b) Use integration by parts to evaluate I . (*Hint:* The integral by parts formula relevant here is

$$\iiint_V f \nabla \cdot \mathbf{A} d\tau = - \iiint_V \nabla f \cdot \mathbf{A} d\tau + \oint_{\partial V} f \mathbf{A} \cdot d\boldsymbol{\sigma},$$

where f is a scalar field, $\mathbf{A}$ is a vector field, V is a 3D volume, and ∂V is its boundary. This formula comes from the product rule $\nabla \cdot (f \mathbf{A}) = \nabla f \cdot \mathbf{A} + f \nabla \cdot \mathbf{A}$ combined with the divergence theorem.)