

## 1 Integrals with the Dirac Delta

Compute the following integrals containing the Dirac delta,

(a)

$$\int_0^{2\pi} \cos(\pi x) \delta(x - 7) dx$$

**Solution:** Since the offset in the argument of the Dirac delta, 7, lies outside of the domain of integration  $[0, 2\pi]$  (note  $2\pi \approx 6.28$ ), the delta function stays zero throughout the domain of the integral. Thus this integral evaluates to **zero**.

(b)

$$\int_0^{2\pi} x^2 \delta(x - 1) dx$$

**Solution:** The offset in the argument of the delta function, 1, is within the domain  $[0, 2\pi]$ , so the result is simply the rest of the integrand evaluated at  $x = 1$ , that is,  $1^2 = 1$ .

(c)

$$\int_{-\infty}^a \delta(x) dx$$

**Solution:** This integral depends on the parameter  $a$ . When  $a < 0$ , the integral is equal to 0, because the domain of the integral does not include the point where  $\delta(x)$  is non-zero. When  $a > 0$ , the integral is 1.

## 2 Delta function with complicated arguments

(a) Show that  $\delta(kx - x_0) = \frac{1}{|k|} \delta(x - x_0/k)$ .

**Solution:** Since delta functions are defined in terms of the integral of  $f(x) \delta(x)$ , we can plug in  $\delta(kx - x_0)$  in place of  $\delta(x)$ . We get the integral

$$\int_{-\infty}^{\infty} f(x) \delta(kx - x_0) dx.$$

If we take the substitution  $u = kx - x_0$ , we get  $x = u/k + x_0/k$  and  $dx = du/|k|$ . Plugging these into the above integral, we get

$$\int_{-\infty}^{\infty} f\left(\frac{u}{k} + \frac{x_0}{k}\right) \delta(u) \frac{1}{|k|} du = \frac{1}{|k|} f\left(\frac{x_0}{k}\right).$$

This gives the same result as the integral of  $f(x)$  multiplied by  $\frac{1}{|k|} \delta(x - x_0/k)$ , which gives us the desired identity.

- (b) i. Express  $\delta(x^2 - a^2)$  in terms of simple  $\delta$  functions.

**Solution:** Since  $x^2 - a^2 = (x - a)(x + a)$ , it has simple zeros at  $a$  and  $-a$ . The derivative of  $g(x) = x^2 - a^2$  is  $g'(x) = 2x$ . Therefore,

$$\delta(x^2 - a^2) = \frac{\delta(x - a)}{|2x|_{x=a}} + \frac{\delta(x + a)}{|2x|_{x=-a}} = \frac{1}{2|a|} (\delta(x - a) + \delta(x + a)).$$

- ii. Compute  $\int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} \delta(\sin(\pi x)) dx$ .

**Solution:** The zeros of  $\sin(\pi x)$  are at all integers  $\dots, -1, 0, 1, \dots$ . These are all simple zeros, because  $|\sin(\pi x)'| = \pi |\cos(\pi x)|$ , and evaluating this at the integers we get  $\pi$ . Therefore,

$$\delta(\sin(\pi x)) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \delta(x - n),$$

and the integral we want is

$$\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2}.$$

The sum of the series is  $\coth(a\pi)/a$ . This series is the partial fraction decomposition of the hyperbolic cotangent function. It can be proved from the Euler product formula for the sin function and differentiating the natural log of that identity.

### 3 Dirac delta and integration by parts

Evaluate the following integral

$$I = \iiint_V (r^2 + R^2) \nabla \cdot \left( \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau$$

where  $V$  is a ball of radius  $R$  centered at the origin.

- (a) **Solution:** Since  $\nabla \cdot \left( \frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi \delta^3(\mathbf{r})$ , and the origin lies within the domain of integration  $V$ , we can evaluate the integral by plugging  $\mathbf{r} = \mathbf{0}$  into the rest of the integrand:

$$\iiint_V (r^2 + R^2) 4\pi \delta^3(\mathbf{r}) d\tau = (0^2 + R^2) \cdot 4\pi = 4\pi R^2.$$

- (b) **Solution:** Applying the integration by parts formula with  $f = r^2 + R^2$  and  $\mathbf{A} = \hat{\mathbf{r}}/r^2$ , we get

$$I = - \iiint_V \nabla(r^2 + R^2) \cdot \frac{\hat{\mathbf{r}}}{r^2} d\tau + \oint_{\partial V} (R^2 + R^2) \frac{\hat{\mathbf{r}}}{R^2} \cdot d\boldsymbol{\sigma}.$$

For the first integral,  $\nabla(r^2 + R^2) = 2r \hat{\mathbf{r}}$ , so  $\nabla(r^2 + R^2) \cdot \hat{\mathbf{r}}/r^2 = 2r/r^2 = 2/r$ . This gives

$$- \int_0^R dr \frac{2}{r} r^2 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi = -4\pi \int_0^R 2r dr = -4\pi R^2.$$

For the surface integral, on the sphere of radius  $R$  we have  $\hat{\mathbf{r}}/R^2 \cdot d\boldsymbol{\sigma} = dA/R^2$ , where  $dA$  is the area element. The total surface area is  $4\pi R^2$ , so the surface integral is

$$(R^2 + R^2) \frac{4\pi R^2}{R^2} = 8\pi R^2.$$

Adding the two contributions,

$$I = -4\pi R^2 + 8\pi R^2 = 4\pi R^2.$$